The Stochastic Transport Equation with Singular Coefficients

by

EMIL HUSTER

THESIS

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Det matematisk- naturvitenskapelige fakultet
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Abstract

This master thesis gives an extensive survey of fundamental concepts in stochastic analysis like Itô's stochastic calculus and Malliavin calculus. These tools form the framework to numerically study the solutions of the stochastic transport equation with singular coefficients. The method used in the numerical simulation is based on the concepts of brackets of stochastic processes introduced in Eisenbaum [1].

Preface

The topic of this master thesis originated from my background in physics and my desire to delve into stochastic analysis. It is the product of my struggles to complete a master of science in statistics.

The topic I write about here is in itself an interesting one. It is well known that ordinary differential equations (ODE's) and partial differential equations (PDE's) in general do not have a unique solution if the coefficient is not Lipschitz continuous. In many cases there might not even be a solution at all for differential equations with such coefficients.

As an example, consider the ODE

$$\frac{dX_t}{dt} = b(t, X_t), \quad X_0 = 0,$$

where $b(t,x) = 2\,\mathrm{sign}(x)\sqrt{|x|}$ is non-Lipschitzian. It is straightforward to check that $X_t = 0$ is a solution for all t. One can also verify that $X_t = \pm t^2$ are solutions. Therefore, the solution in the case of the ODE above is not unique.

However, by superposing the ODE above by a small noise one obtains the corresponding stochastic differential equation (SDE)

$$dX_t = b(t, X_t)dt + \epsilon dB_t, \quad X_0 = x,$$

where $\epsilon > 0$ and B_t is Brownian motion. This SDE is well-posed, meaning that it possesses a unique strong solution $X.$ — no matter how small ϵ is, if b is only bounded and measurable. This strange result was first discovered by Zvonkin [7].

In the case of PDE's, however, the question whether adding noise has a regularising effect is difficult to answer, and only few examples can be found in the literature. Yet, one example is the linear transport equation perturbed by a Brownian motion. It is given by the stochastic partial differential equation (SPDE)

$$\begin{aligned} \partial_t u(t,x) + b(t,x)\partial_x u(t,x) + \partial_x u(t,x) \circ dB_t = 0 \\ u(0,x) = u_0(x), \end{aligned} \tag{$\ast$}$$

where the added noise is in the form of a Stratonovich integral $\partial_x u(t,x) \circ dB_t$ and the coefficient b is bounded and measurable. This SPDE was studied in Mohammed et al. [5] where the authors show that it allows the existence of a Malliavin differentiable unique solution u under certain conditions on the initial data u_0.

The purpose of this master thesis is to present an extensive survey of the basic concepts of stochastic analysis and Malliavin calculus as means to study the SPDE ($\ast$). Also, this thesis contains a numerical study that involves the simulation of solutions to ($\ast$) for non-Lipschitzian coefficients b, as well as a brief study of the convergence of solutions u_ϵ for $\epsilon \to 0$ when $\partial_x u(t,x) \circ dB_t$ is substituted by $\epsilon\partial_x u(t,x) \circ dB_t$.

All definitions, lemmas, propositions and theorems in this thesis are obtained from my references. Hence, I have only provided complete proofs for some of the results where the proofs have been short. In addition, I have made sketches of proofs for the more important theorems, which would require longer proofs. In any case can the complete proofs be found in my references or in references within my references.

Finally, I want to express my gratitude to my partner Iselin who has been most patient in taking care of our new born son Leon while I was occupied with this thesis, Leon who is such a happy and content little boy, my mother Kaja who provided me with valuable guidance as to structuring this thesis and proofread it carefully, and last but not least my supervisor Frank who has patiently answered my naive questions and motivated me when I felt lost and confused.

Emil Huster
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Contents

1 Introduction

Some research on the Internet reveals that the term "transport equation" refers to at least two related expressions. Wikipedia refers to the convection-diffusion equation, also known as the advection-diffusion or drift-diffusion equation. Other sites refer to what Wikipedia calls the advection equation.

The advection equation describes transport of "something" in a fluid by the fluid's bulk motion. This is at least how the English article on Wikipedia.org describes advection; "a transport mechanism of a substance or conserved property by a fluid due to the fluid's bulk motion." An example of this is the transport of pollutants in a river by the flow of bulk water. Advection cannot therefore happen in solids, as it requires currents in the fluid.

Advection is sometimes used as a synonym for convection, but according to some sources on the Internet convection is the combination of advective transport and diffusive transport.

In meteorology, advection is describing the transport of heat, moisture and vorticity by the atmosphere or the ocean. Hence, this is an important concept in weather forecasting.

However, I have noticed that in literature concerning making random perturbations or adding noise to the advection equation, it is referred to as the transport equation. That is the term I will use here, too. It is anyway safe to say that the transport equation shows up in some form or another in many situations in physics where there is some conservation of energy, mass, charge, etc. involved.

During the last few decades, a great effort has been made to study the stochastic transport equation (STE). I refer to the papers by Mohammed et al. [5] and Flandoli et al. [2], and the references therein. More specific, STE's with discontinuous coefficients and driven by noise like Brownian motion, have been an important area of study. My guess is that the STE is so much studied because its solutions is given by the unique strong solutions of the stochastic differential equation (SDE)

$$dX_t^x = b(t, X_t^x)dt + dB_t, \quad t \geq 0, \ X_0^x = x.$$

This master thesis has the following structure:

- In Section 2 I review most of the fundamental theory about stochastic calculus.
- Section 3 is concerned with the basic theory and construction of the Malliavin calculus through the approach of Wiener-Itô chaos expansion.
- Finally, in Section 4 I discuss some results presented in [5] concerning the uniqueness and existence of Malliavin differentiable solutions for the STE with singular coefficients. I also perform a numeric study of the STE with singular coefficients. The algorithm for the simulation is based on concepts of brackets of stochastic processes that were introduced in Eisenbaum [1].

Figure 1.1 gives a visual overview of the most important definitions and theorems in this thesis.

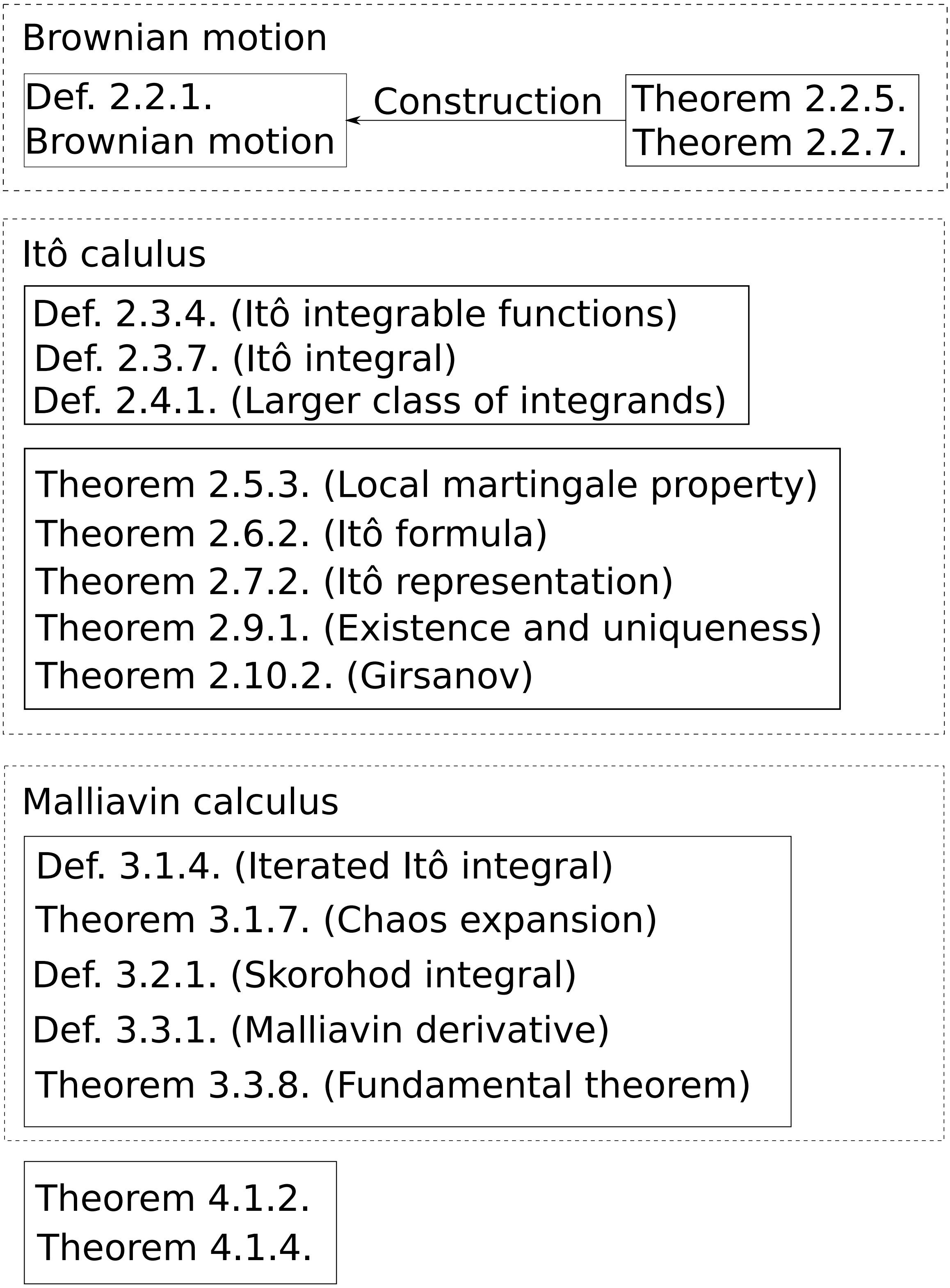


Figure 1.1: An overview of most important theorems in this thesis.

2 Itô's Stochastic Calculus

In this section I will present the theory of Itô's stochastic calculus. It was developed by K. Itô in his 1944 paper, and constitutes the basis for solving stochastic differential equations. One of its most notable applications is to the Black-Scholes theory in finance.

I start by describing some basic concepts that are necessary for the following sections. Then I look at the definition and construction of Brownian motion. This is necessary for the construction of the Itô integral. Further, I will discuss the Itô formula, the Itô representation theorem and the martingale representation theorem. I will present a theorem for existence and uniqueness, and strong and weak solutions of stochastic differential equations. Finally, I will comment on the Girsanov theorem.

2.1 Basic Concepts

To start with I give some basic definitions necessary for the further presentation. In the following, let $(\Omega, \mathcal{F}, P)$ denote a complete probability space. That is, $\mathcal{F}$ is a σ-algebra on Ω and $P : \mathcal{F} \to [0, 1]$ is a probability measure on $(\Omega, \mathcal{F})$. It is complete in the sense that $\mathcal{F}$ contains all subsets of Ω with P-outer measure zero. Let $\mathcal{B}$ denote the Borel σ-algebra on Ω generated by $\mathcal{U}$, the collection of all open subsets of Ω. Then $B \in \mathcal{B}$ are called Borel sets.

Definition 2.1.1. *A random variable is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$. The distribution of X is the probability measure μ_X induced by the random variable and defined by*

$$\mu_X(B) := P(X^{-1}(B)).$$

Definition 2.1.2. *A stochastic process is a parameterised collection of random variables*

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.

The parameter space T in the definition above can be a closed or half-open interval on the real line, i.e. $[a, b]$ or $[0, \infty)$, or even subsets of $\mathbb{R}^n$ for $n \geq 1$. For every fixed $t \in T$ we have a random variable

$$\omega \mapsto X_t(\omega), \quad \omega \in \Omega.$$

Fixing $\omega \in \Omega$, however, gives the path of X_t

$$t \mapsto X_t(\omega), \quad t \in T.$$

Hence, the parameter t is usually interpreted as time, and $X_t(\omega)$ as the position of a particle ω at a given time t.

Kuo [4] defines X_t on the product space $T \times \Omega$ and uses the notation $X(t, \omega)$. With this notation the process can be viewed as a function of two variables

$$X(t, \omega) : T \times \Omega \to \mathbb{R}^n.$$

This is often a convenient interpretation, since it is crucial in stochastic analysis $X(t, \omega)$ being jointly measurable in (t, ω), see Øksendal [8].

2.2 Brownian Motion

The purpose of this section is to briefly treat the mathematical definition and construction of Brownian motion, a type of "noise" which when added turns a differential equation into a stochastic differential equation.

Brownian motion is named after the Scottish botanist Robert Brown, who in 1828 studied pollen grains suspended in water with a microscope. He observed that the pollen grain seemed to move around randomly. It later turned out that the motion was caused by the random bombardment of the water molecules (see page 47 in the book by Karatzas and Shreve [3] and page 11 in the book by Øksendal [8]).

When contemplating the physics behind Brownian motion, it is reasonable that the displacement of the pollen particle is a result of the net force of all the water molecules colliding with it at any instant. Since there are very many water molecules colliding with the pollen particle all the time, the law of large numbers can be applied. Thus it is reasonable for the particle's movement to be described by the normal distribution. Since the water molecules are surrounding the pollen particle (and assumed that the drift is zero), the net force should not have any particular direction. The displacement of the particle from one instant to the next should be influenced by the time that has passed between the sampling of the position. Based on this reasoning one might conclude that the mean displacement is zero and that the variance is some function of the time between succeeding samples of the position. Moreover, the position at one instant should be independent of previous positions. Finally, the particle can not jump from one spot to a another.

This motivates the following definition which is obtained from Kuo [4]:

Definition 2.2.1 (1-dimensional Brownian motion). *A stochastic process $\{B_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P^x)$ is called Brownian motion if it satisfies the conditions:*

1. $P^x(\omega : B_0(\omega) = x) = 1$

2. *For any $0 \leq s < t$, the random variable $B_t - B_s$ is normally distributed with mean x and variance $t - s$, i.e. for $a < b$*

$$P^x(B_t - B_s \in [a, b]) = (2\pi(t-s))^{-1/2} \int_a^b e^{-(y-x)^2/2(t-s)} dy \tag{2.2.1}$$

3. *B_t has independent increments, i.e. for any $0 \leq t_1 < t_2 < \ldots < t_k$, the random variables*

$$B_{t_1},\ B_{t_2} - B_{t_1},\ \cdots,\ B_{t_k} - B_{t_{k-1}}, \tag{2.2.2}$$

are independent.

4. *The sample paths of B_t are continuous for almost all $\omega \in \Omega$, i.e.*

$$P(\omega : B(\cdot, \omega) \text{ is continuous}) = 1 \tag{2.2.3}$$

Following are some properties of Brownian motion. The first one is Proposition 2.2.1. in [4]:

Proposition 2.2.2. *For any $t > 0$, B_t is normally distributed with mean zero and variance t. For any $s, t \geq 0$, $E[B_t B_s] = \min(s, t)$.*

Proof. Since $B_t = B_t - B_0$ it follows from (2) in Definition 2.2.1 that $EB_t = 0$ and $\operatorname{Var} B_t = t$. To show that $E[B_s B_t] = \min(s, t)$ assume first that $s < t$. Then using

$$E[B_s B_t] = E[B_s(B_t - B_s) + B_s^2] = E[B_s(B_t - B_s)] + EB_s^2 = 0 + \operatorname{Var} B_s = s, \quad (2.2.4)$$

where I have used $\operatorname{Var} B_s = EB_s^2 + (EB_s)^2 = EB_s^2$. Similarly for $t < s$. □

The next proposition is Proposition 2.2.3. in [4]:

Proposition 2.2.3 (Translational invariance)**.** *For fixed $t_0 \geq 0$ the stochastic process $\tilde{B}_t = B_{t+t_0} - B_{t_0}$ is also Brownian motion.*

Proof. Since both B_{t+t_0} and B_{t_0} start in zero, $\tilde{B}_t$ also starts in zero, so property (1) is satisfied. Obviously, the same applies for property (4). For any $s < t$,

$$\tilde{B}_t - \tilde{B}_s = B_{t+t_0} - B_{s-t_0}, \quad (2.2.5)$$

so by condition (2) $\tilde{B}_t - \tilde{B}_s$ is normally distributed with mean zero and variance $(t + t_0) - (s + t_0) = t - s$. Hence, $\tilde{B}_t$ satisfies condition (2).

Finally, assume $t_0 > 0$. Then for any $0 \leq t_1 < t_2 < \cdots < t_n$, we have $0 < t_0 \leq t_1 + t_0 < t_2 + t_0 < \cdots < t_n + t_0$. By condition (3), $B(t_k - t_0) - B(t_{k-1} - t_0)$, $k = 1, 2, \ldots, n$, are independent random variables. Hence, by Equation (2.2.5), the random variables $\tilde{B}(t_k) - \tilde{B}(t_{k-1})$ are independent and so $\tilde{B}_t$ satisfies condition (3) in Definition 2.2.1. □

The last proposition before I present the construction of Brownian motion is Proposition 2.2.4. in [4]:

Proposition 2.2.4 (Scaling invariance)**.** *For any real number $\lambda > 0$, the stochastic process $\tilde{B}_t = B_{\lambda t}/\sqrt{\lambda}$ is also a Brownian motion.*

Proof. Conditions (1), (3) and (4) of Definition 2.2.1 are easily verified for the stochastic process $\tilde{B}_t$. Regarding condition (2), note that for any $s < t$,

$$\tilde{B}_t - \tilde{B}_s = \frac{1}{\sqrt{\lambda}}(B_{\lambda t} - B_{\lambda s}), \quad (2.2.6)$$

so $\tilde{B}_t - \tilde{B}_s$ is normally distributed with mean zero and variance $\lambda^{-1}(\lambda t - \lambda s) = t - s$. Hence, $\tilde{B}_t$ satisfies condition (2). □

Construction of Brownian Motion

There are at least three methods for constructing Brownian motion. The first one is due to N. Wiener. The second one is based on Kolmogorov's extension and continuity theorem, and the last one is due to P. Lévy. In the following, I will make account for the second method, using Kolmogorov's theorems.

Now I will explain the construction of 1-dimensional Brownian motion. The n-dimensional construction is not very different and can be found in for instance [8].

I start as in Section 2.2., page 49 in the book by Karatzas and Shreve [3]: Let $\mathbb{R}^T$ denote the set of all real-valued functions on $T = [0,\infty)$. If $F \in \mathcal{B}(\mathbb{R}^n)$, the smallest σ-algebra generated by $\mathbb{R}^n$, then a set on the form

$$\{\omega \in \mathbb{R}^T : (\omega(t_1), \omega(t_2), \ldots, \omega(t_n)) \in F\}, \tag{2.2.7}$$

where $t_i \in T$, $i = 1, \ldots n$, is called an n-dimensional cylinder set in $\mathbb{R}^T$. I will denote the smallest σ-algebra containing all finite-dimensional cylinder sets (2.2.7) in $\mathbb{R}^T$, by $\mathcal{B}(\mathbb{R}^T)$.

Next, the *finite-dimensional distributions* of a stochastic process $\{X_t\}_{t\in T}$ on a probability space $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T), P)$ is defined as the measures

$$\mu_{t_1,\ldots,t_k}(F_1 \times \cdots \times F_k) = P(X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k), \tag{2.2.8}$$

where $F_1, \ldots, F_k \in \mathcal{B}(\mathbb{R}^n)$. See Øksendal [8, page 11].

The family of probability measures $\{\mu_{t_1,\ldots,t_n}\}_{n\geq 1, t_i\in T}$ obtained from different sequences $(t_1, t_2, \ldots, t_n)$, $t_i \in T$, $n \geq 1$, is said to be consistent if the two following conditions are satisfied (see [4, 8]):

Let $F_1, F_2, \ldots, F_n \in \mathcal{B}(\mathbb{R}^n)$.

1. For all $t_1, t_2, \ldots, t_n \in T$, $n \geq 1$ and all permutations σ on $\{1, 2, \ldots, n\}$

$$\mu_{t_{\sigma(1)},\ldots,t_{\sigma(n)}}(F_1 \times \ldots \times F_n) = \mu_{t_1,\ldots,t_n}(F_{\sigma(1)} \times \ldots \times F_{\sigma(n)}), \tag{2.2.9}$$

2. For all $m \geq 1$,

$$\mu_{t_1,\ldots,t_n}(F_1 \times \ldots \times F_n) = \mu_{t_1,\ldots,t_n,t_{n+1},\ldots,t_{n+m}}(F_1 \times \ldots \times F_n \times \mathbb{R} \times \ldots \times \mathbb{R}), \tag{2.2.10}$$

where the set on the right hand side has a total of $k + m$ factors.

Referring to page 50 in Karatzas and Shreve [3], a family of consistent finite-dimensional distributions can be constructed from a given probability measure on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$. The next theorem shows the converse, i.e. that a probability measure on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ can be constructed from a consistent family of finite-dimensional distributions. This provides the facilities for constructing Brownian motion.

Theorem 2.2.5 (Kolmogorov's extension theorem)**.** *Suppose $\{\mu_{t_1,\ldots,t_n}\}_{n\geq 1, t_i\in T}$ is a consistent family of finite-dimensional distributions. Then there exists a probability space $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T), P)$ such that*

$$\mu_{t_1,\ldots,t_k}(F_1 \times \cdots \times F_k) = P(X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k), \tag{2.2.11}$$

for every $t_i \in T$, $n \geq 1$, $F_i \in \mathcal{B}(\mathbb{R}^n)$.

For the proof of this theorem see e.g. [8] and the references therein.

To wrap up the first part of the construction of Brownian motion, the following is my interpretation of [8, 3, 4].

Define a stochastic process

$$B_t(\omega) := \omega(t), \quad t \geq 0,\ \omega \in \mathbb{R}^T$$

on the probability space $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T), P)$.

Then define the Gaussian kernel as

$$p(t,x,y) := (2\pi t)^{-1/2} \exp\left(-\frac{(x-y)^2}{2t}\right), \quad x, y \in \mathbb{R},\ t > 0. \tag{2.2.12}$$

Whenever $0 = t_1 \leq \cdots \leq t_n$, let $\mu_{t_1,\ldots,t_n}(F_1 \times \cdots \times F_n)$ be the finite-dimensional distribution of B_t on $\mathbb{R}^n$, given by

$$\mu_{t_1,\ldots,t_n}(F_1 \times \cdots \times F_n) = \int\limits_{F_1\times\cdots\times F_n} p(t_1, 0, x_1)p(t_2 - t_1, x_1, x_2)\cdots p(t_n - t_{n-1}, x_{n-1}, x_n)dx_1\cdots dx_n. \tag{2.2.13}$$

Here I make use of the convention that $p(0,x,y)dy = \delta_x(y)$, the unit point mass at x (see [8, 4]).

By using Equation (2.2.9) to extend this definition to all finite sequences $t_1, \ldots, t_n$, and knowing that Equation (2.2.10) holds since $\int_{\mathbb{R}} p(t,x,y)dy = 1$ for all $t \geq 0$, it follows by Kolmogorov's extension theorem that Equation (2.2.8) holds for the finite-dimensional distribution $\mu_{t_1,\ldots,t_n}$ of B_t on the measurable space $(\mathbb{R}^T, \mathcal{B}(\mathbb{B}^T))$ with the probability measure P.

The only thing that remains is to show property (4) in Definition 2.2.1, i.e. that the sample paths are continuous. Following [8], I start with the next definition.

Definition 2.2.6. *Suppose $\{X_t\}$ and $\{Y_t\}$ are stochastic processes on $(\Omega, \mathcal{F}, P)$. Then we say that $\{X_t\}$ is a version of $\{Y_t\}$ if*

$$P(\{\omega : X_t(\omega) = Y_t(\omega)\}) = 1 \quad \textit{for all } t. \tag{2.2.14}$$

If X_t is a version of Y_t, then X_t and Y_t have the same finite-dimensional distributions.

Now continuity can be shown with the aid of Kolmogorov's continuity theorem as stated in [8]

Theorem 2.2.7 (Kolmogorov's continuity theorem)**.** *Suppose that the process $X = \{X_t\}_{t\geq 0}$ satisfies the following condition: For all $T > 0$ there exist positive constants α, β, D such that*

$$E[|X_t - X_s|^\alpha] \leq D|t-s|^{1+\beta}, \quad 0 \leq s < t \leq T. \tag{2.2.15}$$

Then there exists a continuous version of X.

With $\alpha = 4, \beta = 1, D = 3$, Equation (2.2.15) holds for Brownian motion. This concludes that Brownian motion exists.

Having established the existence of Brownian motion as defined in Definition 2.2.1, it is now safe to construct an integral involving Brownian motion as the integrator.

2.3 Itô Integrals

In this section I will outline how to define the Itô integral

$$\int_a^b f(t,\omega)dB_t(\omega) \tag{2.3.1}$$

for $0 \leq a < b$. With the definition of this integral, we are one step closer to solve SDE's involving Brownian motion.

The procedure for constructing the Itô integral resembles the procedures for construction of the Riemann-Stieltjes integral and the Lebesgue integral. In order to develop integration with respect to Brownian motion, one first defines the integral of certain simple functions in a Riemann-Stieltjes sense. Then one extends the definition to more general integrands by an approximation procedure, as in the Lebesgue recipe. Central for the approximation procedure is the Itô isometry.

As it is pointed out on page 37 in [4], the motivation behind Itô's theory of stochastic integration can be to acquire a direct method to construct diffusion processes as solutions of SDEs. However, it can also be motivated from the viewpoint of martingales.

As mentioned above, the first thing to do is to define the Itô integral for a class of simple functions. Let us assume that f is a step function and has the form

$$\phi(t,\omega) = \sum_{k\geq 0} e_k(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t), \tag{2.3.2}$$

where $\mathbb{1}$ is the indicator function. The indices

$$t_k = t_{k,n} = \begin{cases} k/2^n & \text{if } a \leq k/2^n \leq b \\ a & \text{if } k/2^n < a \\ b & \text{if } k/2^n > b \end{cases}$$

Subsequently, it is reasonable to define the integral for functions (2.3.2) in the Riemann-Stieltjes sense

$$\int_a^b \phi(t,\omega)dB_t(\omega) := \sum_{k\geq 0} e_k(\omega)[B_{t_{k+1}} - B_{t_k}]. \tag{2.3.3}$$

However, the functions e_k must fulfil some conditions in order for the integral (2.3.3) to be unambiguous. This is seen in [8, Example 3.1.1.] where the computation of the integral (2.3.3) of two simple functions (2.3.2) with $e_k(\omega) = B_{t_k}(\omega)$ in one case and $e_k(\omega) = B_{t_{k+1}}(\omega)$ in the other case shows that the choice of time point $t_k^* \in [t_k, t_{k+1}]$ in the approximation

$$\sum_k f(t_k^*,\omega)[B_{t_{k+1}} - B_{t_k}](\omega)$$

is not arbitrary. In fact, Theorem 4.1.2. in [4] shows that the quadratic variation of Brownian motion, defined as the sum

$$\sum_{k=0}^{n}(B(t_{i+1}) - B(t_i))^2, \qquad \text{is nonzero.}$$

There are two common choices for the points t_k^* mentioned above:

1. $t_k^* = t_k$, i.e. the left end point, leads to the Itô integral

$$\int_a^b f(t,\omega)dB_t(\omega), \quad \text{and}$$

2. $t_k^* = (t_k + t_{k+1})/2$, i.e. the mid point, leads to the Stratonovich integral

$$\int_a^b f(t,\omega) \circ dB_t(\omega).$$

In this context I will use $t_k^* = t_k$. This choice leads to the important consequence that the Itô integral is a martingale, see Definition 2.3.3.

In order to be able to arrive at an unambiguous definition of the stochastic integral, the integrand needs to be part of a special class of functions. I will, however, need to define filtrations and adapted processes first.

Definition 2.3.1 (Filtration). *A filtration on $[0,T]$, $T > 0$, is an increasing (or non-decreasing) family $\{\mathcal{F}_t : t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s < t < \infty$. The filtration $\mathcal{F}_t^X := \sigma\{X_s : 0 \leq s \leq t\}$ generated by a stochastic process X_s, is the smallest σ-algebra for which X_s is measurable with respect to the filtration, for every $s \in [0,t]$. A filtration is said to be right continuous if*

$$\mathcal{F}_t = \bigcap_{n>0}^{\infty} \mathcal{F}_{t+\frac{1}{n}}, \quad \forall\, t \in [a,b).$$

Definition 2.3.2 (Adapted process). *Let $\{\mathcal{N}_t\}_{t\geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $g(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{N}_t$-adapted if for each $t \geq 0$ the function $\omega \to g(t,\omega)$ is $\mathcal{N}_t$-measurable.*

Note that every process X_t is adapted to the filtration $\{\mathcal{F}_t^X\}$.

The next definition is not needed for the construction of the Itô integral, but I still list it here for future reference.

Definition 2.3.3 (Martingale). *A stochastic process $X_t = \{X_t\}_{t\geq 0}$ is called a martingale with respect to the filtration $\mathcal{F}_t$ if*

1. *X_t is $\mathcal{F}_t$-measurable for all t,*

2. *$E[|X_t|] < \infty$ for all t, and*

3. *$E[X_t|\mathcal{F}_s] = X_s$ for all $s \leq t$.*

In the following, let the Brownian motion $B_t(\omega)$ be fixed, and let

$$\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\}$$

be the complete σ-algebra generated by $\{B_s\}_{0\leq s\leq t}$. Then B_t is adapted to the filtration $\{\mathcal{F}_t : t \in [0,T]\}$.

I will now present the three steps involved in the construction of the Itô integral for the following class of functions:

Definition 2.3.4. *Let $L^2_{ad}([a,b] \times \Omega)$ be the class of functions*

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

such that

1. *$(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.*
2. *$f(t,\omega)$ is $\mathcal{F}_t$-adapted.*
3. *$E[\int_a^b f(t,\omega)^2 dt] < \infty$.*

The first step is to construct the Itô integral for step functions $f \in L^2_{\text{ad}}([a,b] \times \Omega)$ given by Equation (2.3.2), i.e.

$$f(t,\omega) = \sum_{i=0}^{n} \xi_i(\omega) \mathbb{1}_{[t_{i-1},t_i)}(t),$$

where ξ_i is $\mathcal{F}_{t_i}$-measurable and $E[\xi_i^2] < \infty$. Now I define

$$I(f) := \sum_{k\geq 0} \xi_k(\omega)[B_{t_{k+1}} - B_{t_k}](\omega).$$

The following lemma is important for the upcoming steps:

Lemma 2.3.5 (Itô isometry)**.**

$$E\left[I(f)^2\right] = \int_a^b E[f^2(t,\omega)]dt \tag{2.3.4}$$

Proof. For the proof, see e.g. [8, 4]. □

To be able to define the Itô integral for general stochastic processes $f \in L^2_{\text{ad}}([a,b]\times\Omega)$, a lemma is needed. This is the second step.

Lemma 2.3.6. *Suppose $f \in L^2_{ad}([a,b]\times\Omega)$. Then there exists a sequence $\{f_n(t) : n \geq 0\}$ of elementary stochastic processes in $L^2_{ad}([a,b] \times \Omega)$ such that*

$$\lim_{n\to\infty} \int_a^b E[|f(t) - f_n(t)|^2]dt = 0. \tag{2.3.5}$$

Proof. See Lemma 4.3.3. in [4]. □

The final step is to use Lemma 2.3.5 and Lemma 2.3.6 to define the Itô integral for general $f \in L^2_{\text{ad}}([a,b] \times \Omega)$. See page 47 in [4] for details.

Definition 2.3.7 (The Itô integral)**.** *Let $f \in L^2_{ad}([a,b] \times \Omega)$. Then the Itô integral of f over the interval $[a,b]$ is defined by*

$$\int_a^b f(t,\omega)dB_t(\omega) := \lim_{n\to\infty} \int_a^b \phi_n(t,\omega)dB_t(\omega), \tag{2.3.6}$$

where the limit is in $L^2(P)$ and $\{\phi_n\}$ is a sequence of step functions such that

$$E\left[\int_a^b (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] \to 0 \quad \text{as } n \to \infty. \tag{2.3.7}$$

As I mentioned in the beginning of this section, the Itô integral is a martingale. This is stated in the following theorem.

Theorem 2.3.8 (Martingale property)**.** *Suppose $f \in L^2_{ad}([a,b]\times\Omega)$. Then the stochastic process*

$$X_t = \int_a^t f(s,\omega)dB_s, \quad a \le t \le b, \tag{2.3.8}$$

is a martingale with respect to the filtration $\{\mathcal{F}_t : a \le t \le b\}$.

Proof. Theorem 4.6.1. in Kuo [4] □

The next important step is to confirm that the Itô integral has continuous sample paths.

Theorem 2.3.9 (Doob martingale inequality)**.** *Let X_t, $a \le t \le b$, be a right continuous martingale. Then for any $\epsilon > 0$,*

$$P(\sup_{a\le t\le b} |X_t| \ge \epsilon) \le \frac{1}{\epsilon} E|X_b|. \tag{2.3.9}$$

Proof. The proof can be found in Theorem 4.5.1. in Kuo [4]. □

Theorem 2.3.10 (Continuity)**.** *Suppose $f \in L^2_{ad}([a,b]\times\Omega)$. Then the stochastic process*

$$X_t = \int_a^b f(s,\omega)dB_s, \quad a \le t \le b,$$

is continuous, that is, almost all of its sample paths are continuous functions on the interval $[a,b]$.

Proof. See Theorem 4.6.2. in Kuo [4] □

This section has established the Itô stochastic integral for a particular class of integrands. It is not a big class, but the functions included there ensures that the Itô integral has the convenient martingale property.

2.4 Expanding the Class of Integrands

I will now show how to extend the class of integrands $L^2_{\text{ad}}([a,b] \times \Omega)$ to a larger class of integrands. This larger class is defined as follows:

Definition 2.4.1. *Let $\mathcal{L}^2([a,b],\Omega)$ denote the class of functions $f(t,\omega)$ fulfilling the following conditions:*

1. *$f(t)$ is adapted to the filtration $\{\mathcal{F}_t\}$,*
2. *$\int_a^b |f(t)|^2 dt < \infty$ almost surely.*

As it is pointed out in [4, Section 5.1]: "Condition (2) means that almost all sample paths are functions in the Hilbert space $L^2[a,b]$. Hence the map $\omega \mapsto f(\cdot,\omega)$ is a measurable function from Ω into $L^2[a,b]$."

To see that $L^2_{\text{ad}}([a,b] \times \Omega)$ is included in $\mathcal{L}^2([a,b],\Omega)$, recall that a function $f(t,\omega)$ is in $L^2_{\text{ad}}([a,b] \times \Omega)$ if it is $\{\mathcal{F}_t\}$-adapted and $\int_a^b E(|f(t)|^2)dt < \infty$. But by the Fubini theorem

$$E\int_a^b |f(t)|^2 dt = \int_a^b E(|f(t)|^2)dt < \infty,$$

so $\int_a^b |f(t)|^2 dt < \infty$ almost surely.

The extension requires a couple of lemmas which I have copied from Kuo [4].

Lemma 2.4.2. *Let $f \in \mathcal{L}^2([a,b],\Omega)$. Then there exists a sequence $\{f_n\}$ in $\mathcal{L}^2([a,b],\Omega)$ such that*

$$\lim_{n\to\infty} \int_a^b |f(t) - f_n(t)|^2 dt = 0$$

almost surely, and hence also in probability.

Proof. See [4, Lemma 5.1.5.]. □

The next lemma is "a key lemma" for achieving the goal of this section.

Lemma 2.4.3. *Let $f(t)$ be a simple stochastic process in $L^2_{ad}([a,b] \times \Omega)$. Then the inequality*

$$P\left(\int_a^b |f(t)dB(t)| > \epsilon\right) \leq \frac{C}{\epsilon^2} + P\left(\int_a^b |f(t)|^2 dt > C\right)$$

holds for any positive constants ϵ and C.

Proof. The proof can be found in Lemma 5.2.2. in [4]. □

Then an approximation lemma is also needed.

Lemma 2.4.4. *Let $f \in \mathcal{L}^2([a,b],\Omega)$. Then there exists a sequence $\{f_n(t)\}$ of simple stochastic processes in $L^2_{ad}([a,b] \times \Omega)$ such that*

$$\lim_{n\to\infty} \int_a^b |f(t) - f_n(t)|^2 dt = 0, \tag{2.4.1}$$

in probability.

Proof. See Lemma 5.3.1. in [4]. $\square$

The tools to define

$$\int_a^b f(t)dB(t) \quad f \in \mathcal{L}^2([a,b],\Omega)$$

are now at hand. By Lemma 2.4.4 there is a sequence $\{f_n(t)\}$ of simple stochastic processes in $L^2_{\text{ad}}([a,b]\times\Omega)$ such that

$$\lim_{n\to\infty}\int_a^b |f(t)-f_n(t)|^2 dt = 0, \quad \text{in probability.}$$

Hence, for every n, the stochastic integral

$$I(f_n) = \int_a^b f_n(t)dB(t)$$

is defined by Definition 2.3.7. Let $f = f_n - f_m$ in Lemma 2.4.3 with $C = \epsilon^3/2$ to get

$$P(|I(f_n) - I(f_m)| > \epsilon) \leq \frac{\epsilon}{2} + P\left(\int_a^b |f_n(t) - f_m(t)|^2 dt > \frac{\epsilon}{2}\right). \tag{2.4.2}$$

The second term satisfies, according to [4], the inequality

$$\begin{aligned} &P\left(\int_a^b |f_n(t) - f_m(t)|^2 dt > \frac{\epsilon}{2}\right) \\ &\qquad \leq P\left(\int_a^b |f(t) - f_n(t)|^2 dt > \frac{\epsilon^3}{8}\right) + P\left(\int_a^b |f(t) - f_m(t)|^2 dt > \frac{\epsilon^3}{8}\right), \end{aligned} \tag{2.4.3}$$

by applying the inequality $|u+v|^2 \leq 2(|u|^2 + |v|^2)$. Using Lemma 2.4.4 on the above inequality, we obtain that

$$\lim_{n,m\to\infty} P\left(\int_a^b |f_n(t) - f_m(t)|^2 dt > \frac{\epsilon^3}{2}\right) = 0.$$

Consequently there is an $N > 1$ such that

$$P\left(\int_a^b |f_n(t) - f_m(t)|^2 dt > \frac{\epsilon^3}{2}\right) < \frac{\epsilon}{2}$$

for all $n, m \geq N$. Reviewing Equation (2.4.2) with regards to the above result, we see that

$$P(|I(f_n) - I(f_m)| > \epsilon) < \epsilon, \quad n, m \geq N.$$

This means that the sequence $\{I(f_n)\}_{n=1}^{\infty}$ converges in probability, and we can define

$$\int_a^b f(t)dB(t) = \lim_{n\to\infty} I(f_n), \quad \text{in probability.}$$

By checking that the limit is independent of the choice of the sequence $\{f_n\}_{n=1}^{\infty}$ one finds that the stochastic integral above is well-defined. In conclusion, we have defined the stochastic integral

$$\int_a^b f(t)dB(t), \quad f \in \mathcal{L}^2([a,b],\Omega).$$

In general, however, the process

$$X_t = \int_a^t f(s,\omega)dB_s(\omega), \quad f \in \mathcal{L}^2([a,b],\Omega),$$

will no longer be a martingale since its expectation is not necessarily finite.

2.5 Local Martingales

The purpose of this section is to provide means in order to mend the loss of the martingale property. The first necessary step is to define stopping times.

Definition 2.5.1 (Stopping time). *A random variable $\tau : \Omega \to [a,b]$ is called a stopping time with respect to a filtration $\{\mathcal{F}_t : a \leq t \leq b\}$ if*

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$$

for all $t \in [a,b]$.

Consider now the process

$$X_t = \int_a^t f(s,\omega)dB_s(\omega) = \int_a^b \mathbb{1}_{[a,t]}(s)f(s,\omega)dB_s(\omega), \quad a \leq t \leq b, \tag{2.5.1}$$

where the integrand belongs to $\mathcal{L}^2([a,b],\Omega)$ for any $t \in [a,b]$.

Let us now define the stochastic process f_n for each n by

$$f_n(t,\omega) = \begin{cases} f(t,\omega), & \text{if } \int_a^t |f(s,\omega)|^2 ds \leq n \\ 0 & \text{otherwise,} \end{cases}$$

and the random variable τ_n by

$$\tau_n(\omega) = \begin{cases} \inf\left\{t : \int_a^t |f(s,\omega)|^2 ds > n\right\}, & \text{if } \{t : \cdots\} \neq \varnothing, \\ b, & \text{if } \{t : \cdots\} = \varnothing. \end{cases}$$

This random variable is a stopping time for each n by [Example 5.4.3. in 4]. By substituting t in Equation (2.5.1) with $t \wedge \tau_n = \min(t, \tau_n)$ and realising that

$$\mathbb{1}_{[a,t\wedge\tau_n(\omega)]}(s)f(s,\omega) = \mathbb{1}_{[a,t]}(s)f_n(s,\omega), \quad \text{for almost all } \omega,$$

we deduct that $f_n \in L^2_{\text{ad}}([a,b] \times \Omega)$. By Theorem 2.3.8 the stochastic process $X_{t\wedge\tau_n}$ is a martingale.

Additionally, the sequence $\{\tau_n\}$ is monotonically increasing and $\tau_n \to b$ almost surely as $n \to \infty$.

Definition 2.5.2 (Local martingale)**.** *An $\{\mathcal{F}_t\}$-adapted stochastic process $X_t, a \leq t \leq b$, is called a local martingale with respect to $\{\mathcal{F}_t\}$ if there exists a sequence of stopping times $\{\tau_n\}_{n=1}^{\infty}$ such that*

1. *τ_n increases monotonically to b almost surely as $n \to \infty$,*
2. *for each n, $X_{t \wedge \tau_n}$ is a martingale with respect to $\{\mathcal{F}_t : a \leq t \leq b\}$.*

The next theorem is stated without proof.

Theorem 2.5.3. *Let $f \in \mathcal{L}^2([a,b],\Omega)$. Then the stochastic process*

$$X_t = \int_a^t f(s,\omega) dB_s(\omega), \quad a \leq t \leq b, \tag{2.5.2}$$

is a local martingale with respect to the filtration $\{\mathcal{F}_t : a \leq t \leq b\}$.

The stochastic process (2.5.2) can be shown to have continuous paths. I refer to Theorem 5.5.5. in [4] for details.

2.6 The Itô Formula

Definition 2.3.7 is not very practical when evaluating stochastic integrals. The Itô formula is therefore an important tool to lighten this task. It can be regarded as corresponding to the chain rule in ordinary calculus.

I will present the general Itô formula as found in [8, 4]. For a more step-wise approach, I can recommend Section 7.1.-7.3. in [4].

To start off, I will use $\mathcal{L}_{\text{ad}}(\Omega, L^1[a,b])$ to denote the class of $\{\mathcal{F}_t\}$-adapted stochastic processes $f(t,\omega)$ that satisfy $\int_a^b |f(t,\omega)| dt < \infty$ almost surely.

Definition 2.6.1 (Itô process in 1 dimension)**.** *Let B_t be 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. A 1-dimensional Itô process is a stochastic process X_t on $(\Omega, \mathcal{F}, P)$ of the form*

$$X_t = X_a + \int_a^t u(s,\omega) ds + \int_a^t v(s,\omega) dB_s, \quad a \leq t \leq b \tag{2.6.1}$$

where $u \in \mathcal{L}_{ad}(\Omega, L^1[a,b])$ and $v \in \mathcal{L}^2([a,b],\Omega)$.

A convenient shorthand for Equation (2.6.1) is

$$dX_t = u(t,\omega) dt + v(t,\omega) dB_t.$$

It should be stressed that this is just a symbolic way of writing (2.6.1), since Brownian paths are nowhere differentiable.

Next I turn to stating the general Itô formula in 1-dimension.

Theorem 2.6.2 (1-dimensional Itô formula). *Let X_t be an Itô process on the form*

$$dX_t = udt + vdB_t.$$

Let $g(t,x) \in C^2([0,\infty),\mathbb{R})$. Then

$$Y_t = g(t,X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dx + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2, \tag{2.6.2}$$

where $(dX_t)^2 = (dX_t)\cdot(dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = 0, \quad dB_t \cdot dB_t = dt. \tag{2.6.3}$$

Proof. A sketch of a proof can be found on on page 46 in [8]. □

2.7 The Itô Representation Theorem

In this section I will give a result named the Itô representation theorem, which will be important for proving the Wiener-Itô chaos expansion Theorem 3.1.7. The following lemma is needed in order to prove the Itô representation theorem. The proof of the lemma can be found on page 50-51 in [8].

Lemma 2.7.1. *The linear span of random variables of the type*

$$\exp\left\{\int_0^T h(t)dB_t - \frac{1}{2}\int_0^T h^2(t)dt\right\}, \quad h \in L^2([0,T]), \tag{2.7.1}$$

is dense in $L^2(\mathcal{F}_T,P)$.

Theorem 2.7.2 (The Itô representation theorem). *Let $F \in L^2(\mathcal{F}_T,P)$. Then there exists a unique stochastic process $f(t,\omega) \in L^2_{ad}([0,T]\times\Omega)$ such that*

$$F(\omega) = E[F] + \int_0^T f(t,\omega)dB(t). \tag{2.7.2}$$

Sketch of proof: The idea is to show that Equation (2.7.2) holds for F on the form (2.7.1) and then to use Lemma 2.7.1 to approximate arbitrary $F \in L^2(\mathcal{F}_T,P)$ with linear combinations of functions F_n on the form (2.7.1).

Let

$$Y_t(\omega) = \exp\left\{\int_0^t h(t)dB_t - \frac{1}{2}\int_0^t h^2(t)dt\right\}, \quad 0 \le t \le T$$

and use Itô's formula to show that (2.7.2) holds for $F = Y_T$. By using the Itô isometry, it can be shown that $F = \lim_{n\to\infty} F_n$ where the limit is taken in $L^2(\mathcal{F}_T,P)$.

Uniqueness also follows from the Itô isometry. □

2.8 The Martingale Representation Theorem

The next theorem is useful in mathematical finance. The proof found in [8, Theorem 4.3.4] is based on the Itô representation theorem.

Theorem 2.8.1 (The martingale representation theorem)**.** *Let $B(t) = (B(t_1), \ldots, B(t_n))$ be n-dimensional Brownian motion. Suppose M_t is an $\mathcal{F}_t$-martingale w.r.t. P and that $M_t \in L^2(P)$ for all $t \geq 0$. Then there exists a unique stochastic process $g(s, \omega)$ such that $g \in \mathcal{L}^2([a, b], \Omega)$ for all $t \geq 0$ and*

$$M_t(\omega) = E[M_0] + \int_0^t g(s, \omega) dB(s) \qquad \text{a.s. for all } t \geq 0.$$

2.9 Existence and Uniqueness of Stochastic Differential Equations

The following theorem provides sufficient conditions for existence and uniqueness of stochastic differential equations.

Theorem 2.9.1. *Let $T > 0$ and $b(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma(\cdot, \cdot) : [0, T] \times \mathbb{R}^{n \times m}$ be measurable functions satisfying*

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|), \quad x \in \mathbb{R}^n,\ t \in [0, T] \tag{2.9.1}$$

for some constant C, and such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|, \quad x, y \in \mathbb{R}^n,\ t \in [0, T] \tag{2.9.2}$$

for some constant D. Here $|\sigma|^2 = \sum |\sigma_{ij}|^2$. Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^m$ generated by $B_s(\cdot), s \geq 0$ such that

$$E[|Z|^2] < \infty.$$

Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T,\ X_0 = Z \tag{2.9.3}$$

has a unique t-continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s(\cdot)$, $s \leq t$ and

$$E\left[\int_0^T |X_t|^2 dt\right] < \infty.$$

Proof. I refer to Theorem 5.2.1. in [8] for the proof. □

As is pointed out in the remark following Theorem 5.2.1. in [8], the condition (2.9.1) is imposed to prevent the solution to explode. Condition (2.9.2) guarantees that a solution is unique.

2.10 Girsanov Theorem

When dealing with Brownian motion in the previous sections, it has been tacit that this is meant with respect to the probability measure P. It might be natural to ask the following two questions:

1. Given a stochastic process, is it possible to determine if it is a Brownian motion with respect to a certain probability measure?
2. If so, how does one obtain this specific probability measure?

The first question is answered by the following

Theorem 2.10.1 (The Lévy characterization of Brownian Motion)**.** *Let X_t be a real continuous stochastic process on a probability space $(\Omega, \mathcal{H}, Q)$. Then the following 1. and 2. are equivalent:*

1. X_t is a Brownian motion with respect to Q and

2. a) X_t is a martingale with respect to Q and

b) $X_t^2 - t$ is a martingale with respect to Q.

Proof. The proof can be found in Theorem 3.3.16. [3] or Theorem 8.4.2. [4]. □

In the following, let $E = E_P$ denote the expectation with respect to the probability measure P. The second question is answered by

Theorem 2.10.2 (The Girsanov theorem)**.** *Let $X_t,\ t \in [0, T]$ be a an Itô process on the form*

$$dX_t = u(t, \omega)dt + dB_t. \tag{2.10.1}$$

Assume

$$M_t := \exp\left(-\int_0^t u(s, \omega)dB(s) - \frac{1}{2}\int_0^t a^2(s, \omega)ds\right), \quad 0 \le t \le T \tag{2.10.2}$$

is a martingale with respect to $\{\mathcal{F}_t\}$ and P, i.e. the Novikov condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T a^2(s, \omega)ds\right)\right] < \infty, \tag{2.10.3}$$

is satisfied. Define the measure Q on $\mathcal{F}_T$ by

$$dQ(\omega) := M_T(\omega)dP(\omega). \tag{2.10.4}$$

Then Q is a probability measure on $\mathcal{F}_T$ and X_t is a Brownian motion with respect to Q for $0 \le t \le T$.

Proof. The proof of this theorem and of two other versions can be found in [8, Section 8.6.]. □

The Girsanov theorem is an important result in general probability theory and hence in stochastic analysis. It has also many applications as for example determining the most probable price process for a certain stock.

3 Malliavin Calculus

Malliavin calculus provides the framework for differentiating random variables. It was invented by Paul Malliavin in the 1970's. Since then it has found many applications. In particular in mathematical finance, where it is used to compute price sensitivities known as Greeks (because of Greek letters, not bad economic politics).

In Section 3.1 I will describe the construction of multiple Wiener-Itô integrals and the Wiener-Itô chaos expansion. This last result will be fundamental for the development of the Malliavin calculus.

In Section 3.2 I will give the definition of the Skorohod integral. In Section 3.3 I will treat the Malliavin derivative. And finally, in Section 3.4 I will give an important result from the efforts of the previous sections.

3.1 Wiener-Itô Chaos Expansion

The expression "chaos expansion" that I have adopted from [6], is probably rooted in N. Wieners definitions of polynomial and homogeneous chaos in his study of statistical mechanics in 1938. Polynomial chaos was defined by him as sums of finitely many multiple integrals with respect to Brownian motion. They are not orthogonal for different order, but the homogeneous chaoses are orthogonal. However, it was K. Itô who in 1951 introduced multiple integrals that turn out to be homogeneous chaos.

I will start this exploration by giving a summary of Sections 9.4.-9.8. in the book [4]. Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space and let B_t be a Brownian motion with respect to P. I will refer to the integral of a deterministic function $f \in L^2[a,b]$ over a fixed, finite interval $[a,b]$ with respect to Brownian motion, $I(f) = \int_a^b f(t)dB(t)$, as a Wiener integral. This Wiener integral is measurable with respect to the Brownian σ-algebra

$$\mathcal{F}^B := \sigma\{B_t : a \le t \le b\} \subset \mathcal{F}.$$

A product $I(f_1)I(F_2)\cdots I(f_k)$ with $f_1, f_2, \ldots, f_k \in L^2[a,b]$, is referred to as polynomial chaos.

Let $L^2_B(\Omega)$ denote the Hilbert space of P-square integrable functions on Ω that are measurable with respect to $\mathcal{F}^B$. The goal is to find an expansion of the functions $f \in L^2_B(\Omega)$ in terms of an orthonormal basis on the probability space $(\Omega, \mathcal{F}^B, P)$.

Put $J_0 = \mathbb{R}$ and for $n \in \mathbb{N}$, define J_n to be the $L^2_B(\Omega)$-closure of the linear space spanned by constant functions and polynomial chaos of degree $\le n$. Then

$$J_0 \subset J_1 \subset \cdots \subset L^2_B(\Omega).$$

Next, let $K_0 = \mathbb{R}$ and for each $n \in \mathbb{N}$, let K_n be the orthogonal complement of J_{n-1} in J_n, i.e.

$$J_n = J_{n-1} \oplus K_n.$$

Then

$$K_0, K_1, \ldots, K_n, \ldots$$

is a sequence of orthogonal sub-spaces of the real Hilbert space $L^2_B(\Omega)$, and each element K_n is referred to as homogeneous chaos of order n.

It can now be shown that every function $\phi \in L^2_B(\Omega)$ has a unique homogeneous chaos expansion ([4, Theorem 9.4.7.]), and that the projection of $L^2_B(\Omega)$ onto K_n of Wiener integrals of nonzero orthogonal functions is given by a product of Hermite functions ([4, Theorem 9.4.9.]).

To find an orthonormal basis for $L^2_B(\Omega)$ that works for series expansion of functions in $L^2_B(\Omega)$, let $\{e_k\}_{k=1}^{\infty}$ be a fixed orthonormal basis for $L^2[a,b]$. Define

$$\mathcal{H}_{n_1,n_2,\ldots} = \prod_k \frac{1}{\sqrt{n_k!}} H_{n_k}(I(e_k)),$$

where $\{n_k\}_{k=1}^{\infty}$ is a non-negative sequence of integers with finite sum and $H_n(x)$ denotes the Hermite polynomial in x of order n with parameter 1.

Theorem 9.5.4. in [4] shows that for any fixed $n \in \mathbb{N}$, the collection of functions

$$\{\mathcal{H}_{n_1,n_2,\ldots} : n_1 + n_2 + \cdots = n\} \tag{3.1.1}$$

is an orthonormal basis for the space K_n. This leads to the result that Equation (3.1.1) is an orthonormal basis for the Hilbert space $L^2_B(\Omega)$, and that every function in $L^2_B(\Omega)$ can be represented as a unique series expansion in terms of the functions in (3.1.1).

As it turns out, the multiple Wiener-Itô integral,

$$\int_{T^n} f(t_1, t_2, \ldots, t_n) dB(t_1) dB(t_2) \cdots dB(t_n),$$

where $T = [a,b]$, has a one-to-one correspondence with the homogeneous chaoses of order n.

Construction of the Wiener-Itô integral

The idea behind the construction of the multiple Wiener-Itô integral is to define the integral in terms of off-diagonal step functions, and then to approximate the functions in $L^2(T^n)$ that we want to integrate, by off-diagonal step functions.

Let $a = \tau_0 < \tau_1 < \cdots < \tau_k = b$. An off-diagonal step function in T^n is a step function on the form

$$f = \sum_{1 \le i_1, \ldots, i_n \le k} a_{i_1,\ldots,i_n} \mathbb{1}_{[\tau_{i_1 - 1}, \tau_{i_1}) \times \cdots \times [\tau_{i_n - 1}, \tau_{i_n})}, \tag{3.1.2}$$

where the coefficients vanish on the diagonal set

$$D = \{(t_1, t_2, \ldots, t_n) \in T^n : t_i = t_j \text{ for some } i \neq j\},$$

i.e.

$$a_{i_1,\ldots,i_n} = 0 \text{ if } i_p = i_q \text{ for some } p \neq q. \tag{3.1.3}$$

For a function f given by Equation (3.1.2), define the integral

$$I_n(f) := \sum_{1 \le i_1, \dots, i_n \le k} a_{i_1, \dots, i_n} \xi_{i_1} \cdots \xi_{i_n}, \tag{3.1.4}$$

where $\xi_{i_m} := B(\tau_{i_m}) - B(\tau_{i_m - 1})$, $1 \le m \le n$.

Definition 3.1.1 (Symmetric function)**.** *A real function $g : T^n \to \mathbb{R}$ is called symmetric if*

$$g(t_{\sigma_1}, \dots, t_{\sigma_n}) = g(t_1, \dots, t_n) \tag{3.1.5}$$

for all permutations $\sigma = (\sigma_1, \dots, \sigma_n)$ of $(1, 2, \dots, n)$.

For a function f, I will use $\tilde{f}$ to denote its symmetrisation, and $\tilde{L}^2(T^n) \subset L^2(T^n)$ denote the space of symmetric square integrable Borel functions on T^n.

The integral of an off-diagonal step function f defined by Equation (3.1.4), can be shown to be equal to the integral of its symmetrisation $\tilde{f}$, i.e. $I_n(f) = I_n(\tilde{f})$ (see [4, Lemma 9.6.2]). As for the Itô integral, the expectation $E[I_n(f)] = 0$ and it possesses an isometry ([4, Lemma 9.6.3.])

$$E[I_n(f)^2] = n! \int_{T^n} |\tilde{f}(t_1, \dots, t_n)|^2 dt_1 \cdots dt_n. \tag{3.1.6}$$

Finally, it can be shown that for a function in $L^2(T^n)$ there exists a sequence $\{f_k\}$ of off-diagonal step functions that converge to f point-wise.

Definition 3.1.2 (Multiple Wiener-Itô integral)**.** *Let $f \in L^2(T^n)$. Then we define the multiple Wiener-Itô integral*

$$I_n(f) := \int_{T^n} f(t_1, \dots, t_n) dB(t_1) \cdots dB(t_n). \tag{3.1.7}$$

Note that in [6] this integral, as well as the integral in Definition 3.1.4, are called n-fold iterated Itô integrals.

This integral has some nice properties. Let the norm on $L^2(T^n)$ be given by

$$||f||^2_{L^2(T^n)} := \int_{T^n} f^2(t_1, \dots, t_n) dt_1 \cdots dt_n.$$

Proposition 3.1.3. *Let $f \in L^2(T^n)$, $n \ge 1$. Then*

1. *$I_n(f) = I_n(\tilde{f})$, where $\tilde{f}$ is the symmetrisation of f.*
2. *$E[I_n(f)] = 0$*
3. *$E[I_n(f)^2] = n! \, ||\tilde{f}||^2_{L^2(T^n)}$.*

Next, I give a result for easing the computational burden involved in evaluating the multiple Wiener-Itô integral. Let the set

$$S_n = \{(t_1, \dots, t_n) \in [0,T]^n : 0 \leq t_1 \leq \cdots \leq t_n \leq T\}.$$

The set S_n occupies $1/!\,n$-th fraction of the n-dimensional cube T^n. Provided that $g \in \tilde{L}^2(T^n)$ then the norm induced by $L^2(T^n)$ on $L^2(S_n)$, the space of square integrable functions on S_n, is given by

$$||g||^2_{L^2(T^n)} = n! \int_{S_n} g^2(t_1, \dots, t_n) dt_1 \cdots dt_n = n!\, ||g||^2_{L^2(S_n)}. \tag{3.1.8}$$

Definition 3.1.4 (Iterated Itô integral)**.** *Let f be a deterministic function defined on S_n $(n \geq 1)$ such that*

$$||f||^2_{L^2(S_n)} := \int_{S_n} f^2(t_1, \dots, t_n) dt_1 \cdots dt_n < \infty,$$

Then we can define the n-fold iterated Itô integral as

$$J_n(f) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \dots, t_n) dB(t_1) \cdots dB(t_n). \tag{3.1.9}$$

Theorem 3.1.5. *Let $f \in L^2(T^n)$. Then*

$$I_n(f) = n!\, J_n(\tilde{f}),$$

where $\tilde{f}$ is the symmetrisation of f.

Proof. The proof can be found in [4, page 173]. □

The next result is a formula for computing the iterated Itô integral, see [6, page 10].

Proposition 3.1.6. *Let $f_1, \dots, f_k$ be nonzero orthonormal functions in $L^2([a,b])$ and $n_1, \dots, n_k$ be positive integers. Then we have that*

$$I_n(f_1^{\otimes n_1} \otimes \cdots \otimes f_k^{\otimes n_k}) = \prod_i^k H_{n_i}(I(f_i)). \tag{3.1.10}$$

In particular, for any nonzero $f \in L^2([a,b])$,

$$I_n(f) = ||f||^n_{L^2([a,b])} H_n\left(\frac{I(f)}{||f||_{L^2([a,b])}}\right). \tag{3.1.11}$$

Finally, the most important theorem of this section, which will be the basis for defining the Skorohod integral and the Malliavin derivative.

Theorem 3.1.7 (The Wiener-Itô chaos expansion)**.** *Let ξ be and $\mathcal{F}_T$-measurable random variable in $L^2(P)$. Then there exists a unique sequence $\{f\}_{n=0}^{\infty}$ of functions $f_n \in \tilde{L}^2([0,T]^n)$ such that*

$$\xi = \sum_{n=0}^{\infty} I_n(f_n), \tag{3.1.12}$$

where the convergence is in $L^2(P)$. Moreover, we have the isometry

$$||f||^2_{L^2(P)} = \sum_{n=0}^{\infty} n!\, ||f_n||^2_{L^2([0,T]^2)}. \tag{3.1.13}$$

Sketch of proof: Use the Itô representation theorem to write

$$\xi = E[\xi] + \int_0^T \phi_1(s_1) dB(s_1), \tag{3.1.14}$$

where $\phi(s_1)$, $0 \leq s_1 \leq T$, is $\{\mathcal{F}_t\}$-adapted such that

$$E\left[\int_0^T \phi^2(s_1) ds_1\right] \leq E[\xi^2]. \tag{3.1.15}$$

Apply the Itô isometry again to $\{\mathcal{F}_t\}$-adapted processes $\phi_1(s_1), \phi_2(s_2, s_1), \ldots,$ $\phi_{n+1}(s_{n+1}, s_n, \ldots, s_1)$ for $0 \leq s_{n+1} \leq s_n \leq \cdots \leq s_1 \leq T$. Define

$$\begin{aligned} g_0 &= E[\xi], \\ g_1(s_1) &= E[\phi_1(s_1)], \\ g_2(s_2, s_1) &= E[\phi_2(s_2, s_1)], \\ &\ldots, \\ g_{n+1}(s_{n+1}, \ldots, s_1) &= E[\phi_1(s_{n+1}, \ldots, s_1)]. \end{aligned}$$

Then after n steps

$$\xi = \sum_{k=0}^{n} J_k(g_k) + \int_{S_{n+1}} \phi_{n+1} dB^{\otimes(n+1)}.$$

The expression

$$\int_{S_{n+1}} \phi_{n+1} dB^{\otimes(n+1)} := \int_0^T \int_0^{t_{n+1}} \cdots \int_0^{t_2} \phi_{n+1}(t_1, \ldots, t_{n+1}) dB(t_1) \cdots dB(t_{n+1})$$

is the $(n+1)$-fold iterated integral of ψ_{n+1}.

The second part of the sum above converges to zero, and by putting

$$g_n(t_1, \ldots, t_n) = 0, \quad (t_1, \ldots, t_n) \in [0,T]\ S_n,$$

we have extended g_n to $[0,T]$. Now define $f_n := \tilde{g}_n$ to be the symmetrisation of g_n. Then we have

$$I_n(f_n) = n!\, J_n(f_n) = n!\, J(\tilde{g}_n) = J_n(g_n).$$

$\square$

3.2 The Skorohod Integral

The stochastic integral to be covered in this section was introduced by A. Skorohod in 1975. It is of tremendous importance in the theory of stochastic processes. This is partly because it unifies several concepts.

It can be seen as an extension of the Itô integral to integrands that not necessarily are adapted to the filtration $\{\mathcal{F}_t\}$. It is the adjoint of the Malliavin derivative to be covered in the next section. And finally, it is an infinite-dimensional generalisation of the divergence operator from classical vector calculus.

Let $u = u(t,\omega),\ t \in [a,b],\ \omega \in \Omega$, be a measurable stochastic process such that $u(t)$ is a $\mathcal{F}_T$-measurable random variable in $L^2(P)$ for all $t \in [a,b]$. Then for each $t \in [a,b]$ there are symmetric functions $f_{n,t} = f_{n,t}(t_1,\dots,t_n),\ (t_1,\dots,t_n) \in [a,b]^n$ in $\tilde{L}^2([a,b]^n),\ n \in \mathbb{N}$ such that u has the chaos expansion

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}).$$

The functions $f_{n,t},\ n \in \mathbb{N}$, depend on a parameter $t \in [a,b]$. Hence, f_n can be regarded as a function of $n+1$ variables

$$f_n(t_1,\dots,t_n,t_{n+1}) = f_n(t_1,\dots,t_n,t) \coloneqq f_{n,t}(t_1,\dots,t_n).$$

The symmetrisation $\tilde{f}_n$ of f_n is given by

$$\begin{aligned}\tilde{f}_n(t_1,\dots,t_n,t_{n+1}) = \frac{1}{n+1}[&f_n(t_1,\dots,t_n,t_{n+1}) \\ &+ f_n(t_2,\dots,t_{n+1},t_1) + \cdots + f_n(t_1,\dots,t_{n+1},t_n)].\end{aligned} \tag{3.2.1}$$

Definition 3.2.1 (Skorohod integral)**.** *Let $u(t),\ t \in [a,b]$, be a measurable stochastic process such that for all $t \in [a,b]$ the random variable $u(t)$ is $\mathcal{F}_T$-measurable and $E[\int_a^b u^2(t)dt] < \infty$. Let its Wiener-Itô chaos expansion be*

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)).$$

Then we define the Skorohod integral of u by

$$\delta(u) \coloneqq \int_a^b u(t)\delta B(t) \coloneqq \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \tag{3.2.2}$$

when convergent in $L^2(P)$, and we say that u is Skorohod integrable.
The $\tilde{f}_n,\ n = 1,2,\dots$, are the symmetric functions derived from $f_n(\cdot,t),\ n = 1,2,\dots$.

Remark 3.2.2. *A stochastic process is Skorohod integrable if and only if*

$$E[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)!\,||\tilde{f}_n||^2_{L^2([0,T]^{n+1})}.$$

The following theorem confirms the statement that the Skorohod integral is an extension of the Itô integral.

Theorem 3.2.3. *Let $u = u(t), t \in [a,b]$, be a measurable $\{\mathcal{F}_t\}$-adapted stochastic process in $L^2([a,b])$. Then u is Skorohod integrable and its Skorohod integral coincides with the Itô integral*

$$\int_a^b u(t)\delta B(t) = \int_a^b u(t)dB(t). \tag{3.2.3}$$

3.3 The Malliavin Derivative

There are several ways of constructing the Malliavin derivative. One way is to perform it on the Wiener space. See Appendix A in [6]. Here, instead, I will follow Chapter 3 in Nunno et al. [6] where construction is based on the chaos expansion. Another way is to construct the Malliavin derivative as a stochastic gradient on the space of tempered distributions $\Omega = S'(\mathbb{R})$, see Chapter 6 in [6].

Definition 3.3.1. *Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable with chaos expansion*

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in \tilde{L}^2([0,T]^n)$, $n = 1, 2, \ldots$ are symmetric functions. Then we say that $F \in \mathbb{D}_{1,2}$ if

$$||F||^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} nn!\, ||f_n||^2_{L^2([0,T]^n)} < \infty. \tag{3.3.1}$$

If $F \in \mathbb{D}_{1,2}$, define the Malliavin derivative D_tF of F at time t as the expansion

$$D_tF := \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t)), \qquad t \in [0,T]. \tag{3.3.2}$$

The next result is fundamental.

Theorem 3.3.2 (Closability of the Malliavin derivative.)**.** *Suppose $F \in L^2(P)$ and $F_k \in \mathbb{D}_{1,2}$, $k = 1, 2, \ldots$, such that*

- *$F_k \to F$, $k \to \infty$, in $L^2(P)$*
- *$\{D_tF_k\}_{k=1}^{\infty}$ converges in $L^2(P \times \lambda)$.*

Then $F \in \mathbb{D}_{1,2}$ and $D_tF_k \to D_tF$, $k \to \infty$, in $L^2(P \times \lambda)$

The next two theorems are analogues to the results in real calculus with the same names.

Theorem 3.3.3 (Product rule.). *Suppose $F_1, F_2 \in \mathbb{D}^0_{1,2}$, i.e. they have chaos expansions with finitely many terms. Then $F_1, F_2 \in \mathbb{D}_{1,2}$ and the product $F_1F_2 \in \mathbb{D}_{1,2}$ with*

$$D_t(F_1F_2) = F_1D_tF_2 + F_2D_tF_1. \tag{3.3.3}$$

Theorem 3.3.4 (Chain rule). *Let $G \in \mathbb{D}_{1,2}$ and $g \in C^1(\mathbb{R})$ with bounded derivative. Then $g(G) \in \mathbb{D}_{1,2}$ and*

$$D_tg(G) = g'(G)D_tG. \tag{3.3.4}$$

Next are some results regarding the relationships between the Malliavin derivative and the Skorohod integral. The first theorem shows that the Malliavin derivative is the adjoint operator of the Skorohod integral.

Theorem 3.3.5 (Duality formula.). *Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable and let u be a Skorohod integrable stochastic process. Then*

$$E\left[F\int_0^T u(t)\delta B(t)\right] = E\left[\int_0^T u(t)D_tFdt\right]. \tag{3.3.5}$$

Theorem 3.3.6 (Integration by parts.). *Let $u(t)$, $t \in [0,T]$, be a Skorohod integrable stochastic process and $F \in \mathbb{D}_{1,2}$ such that the product $Fu(t)$, $t \in [0,T]$, is Skorohod integrable. Then*

$$F\int_0^T u(t)\delta B(t) = \int_0^T Fu(t)\delta B(t) + \int_0^T u(t)D_tFdt. \tag{3.3.6}$$

The next result is proven with the aid of the duality formula Theorem 3.3.5.

Theorem 3.3.7 (Closability of the Skorohod integral.). *Suppose that $u_n(t)$, $t \in [0,T]$, $n = 1, 2, \ldots,$ is a sequence of Skorohod integrable stochastic processes and that the corresponding sequence of Skorohod integrals*

$$\delta(u_n) := \int_0^T u_n(t)\delta B(t), \quad n = 1, 2, \ldots$$

converges in $L^2(P)$. Also suppose that

$$\lim_{n\to\infty} u_n = 0, \quad \text{in } L^2(P \times \lambda).$$

Then

$$\lim_{n\to\infty} \delta(u_n) = 0, \quad \text{in } L^2(P).$$

The last result of this section gives a useful connection between differentiation and integration.

Theorem 3.3.8 (The fundamental theorem.). *Let $u(s)$, $s \in [0,T]$, be a stochastic process such that*

$$E\left[\int_0^T u^2(s)ds\right] < \infty$$

and assume that for all $s \in [0,T]$, $u(s) \in \mathbb{D}_{1,2}$ and that for all $t \in [0,T]$, $D_t u$ is Skorohod integrable. Moreover, assume

$$E\left[\int_0^T (\delta(D_t u))^2 ds\right] < \infty.$$

Then $\int_0^T u(s)\delta B(s)$ is well-defined and belongs to $\mathbb{D}_{1,2}$ and

$$D_t\left(\int_0^T u(s)\delta B(s)\right) = \int_0^T D_t u(s)\delta B(s) + u(t). \tag{3.3.7}$$

3.4 The Clark-Ocone Formula

This section deals with the explicit representation of the integrand in the Itô integral with regard to the Malliavin derivative. The following results are applicable to problems in mathematical finance like hedging and sensitivity analysis.

Theorem 3.4.1 (The Clark-Ocone formula.). *Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable. Then*

$$F = E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dB(t). \tag{3.4.1}$$

The proof of the next theorem, which can be found on page 47 in [6], uses several features of the Malliavin derivative and the Skorohod integral. This theorem can be used to prove the Black-Scholes formula, see Example 4.11. in [6].

Theorem 3.4.2 (The Clark-Ocone formula under change of measure.). *Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable. Suppose that*

$$E_Q[|F|] < \infty \tag{3.4.2}$$

$$E_Q\left[\int_0^T |D_t F|^2 dt\right] < \infty. \tag{3.4.3}$$

Also assume that $u(s) \in \mathbb{D}_{1,2}$ for almost all $Z(T)F \in \mathbb{D}_{1,2}$ and

$$E_Q\left[|F|\int_0^T \left(\int_0^T D_t u(s)dB(t) + \int_0^T u(s)D_t u(s)ds\right)^2 dt\right] < \infty. \tag{3.4.4}$$

Then

$$F = E_Q[F] + \int_0^T E_Q\left[(D_t F - F\int_t^T D_t u(s)d\tilde{B}(s))|\mathcal{F}_t\right] d\tilde{B}(t). \tag{3.4.5}$$

4 Application to the Stochastic Transport Equation

From the theory of ordinary differential equations (ODE's) it is known that an ODE has a unique global solution if the coefficient fulfils the linear growth condition and the Lipschitz condition. However, by perturbing the ODE with a small noise, e.g. by $\epsilon > 0$ times a Brownian motion B_t, it turns out that the resulting stochastic differential equation (SDE) possesses a strong solution as long as the coefficient is bounded and measurable. See Zvonkin [7]. This seems to be the case no matter how small ϵ is.

The next question is whether the regularising effect that noise seems to have on ODE's with "bad" coefficients can be found for partial differential equations (PDEs), too. The linear transport equation modified by a Brownian multiplicative noise has been studied with regards to this question in several papers, e.g. Flandoli et al. [2] and Mohammed et al. [5].

The latter paper offers new contributions to the existing theory of SDE's. Requiring only that the drift vector is bounded and measurable, Mohammed et al. shows well-posedness of the corresponding singular SDE. Moreover, they construct a unique stochastic flow for the singular SDE which further is utilised to generate a unique differentiable solution to the stochastic transport equation (STE) with a bounded measurable drift coefficient. This is quite remarkable, since the corresponding deterministic transport equation is ill-posed in general.

4.1 Construction of Solutions

In this section I outline how the tools discussed in the previous sections are used to obtain unique solutions of the STE with merely bounded measurable coefficients. More precisely, I will present some of the results from [5] obtained by coupling Malliavin calculus with new probabilistic estimates.

The first part is to review the analysis of the spatial regularity in the initial condition $x \in \mathbb{R}^d$ for strong solutions $X^x_\cdot$ to the d-dimensional SDE

$$X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x})du + B_t - B_s, \quad s, t \in \mathbb{R}. \tag{4.1.1}$$

The drift coefficient $b : \mathbb{B} \times \mathbb{R}^d \to \mathbb{R}^d$ in the SDE above is only Borel measurable and bounded, and the SDE is driven by standard Brownian motion $B_\cdot$ in $\mathbb{R}^d$. The above SDE is known to have a unique strong global solution $X_\cdot^{s,x}$ for each $x \in \mathbb{R}^d$.

One of the main objectives is to develop a new method for constructing a Sobolev differentiable stochastic flow for (4.1.1). The strategy used in [5] makes use of ideas from Malliavin calculus combined with new, difficult probabilistic estimates on the spatial weak derivatives of solutions of the SDE. A remarkable aspect of these estimates is that they are independent of the spatial regularity of the drift coefficient b.

The second part is to utilise the existence of a Sobolev differentiable stochastic flow for the SDE (4.1.1) to acquire a unique weak solution $u(t,x)$ of the STE

$$d_t u(t,x) + (b(t,x)\cdot Du(t,x))dt + \sum_{i=1}^{d} e_i \cdot Du(t,x) \circ dB_t^i = 0 \qquad u(0,x) = u_0(x), \tag{4.1.2}$$

where b is only bounded and measurable, $u_0 \in C_b^1(\mathbb{R}^d)$ and $\{e_i\}_{i=1}^d$ a basis for $\mathbb{R}^d$. Recall that the corresponding deterministic transport equation is in general ill-posed. Hence, this is an interesting result.

The following definition is Definition 1 in [5].

Definition 4.1.1. *A map $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d \ni (s,t,x,\omega) \mapsto \phi_{s,t}(x,\omega) \in \mathbb{R}^d$ is a stochastic flow of homeomorphisms for the SDE* (4.1.1) *if there exists a universal set $\Omega^* \in \mathcal{F}$ of full Wiener measure such that for all $\omega \in \Omega^*$, the following statements are true:*

1. *For any $x \in \mathbb{R}^d$, the process $\phi_{s,t}(x,\omega)$, $s,t \in \mathbb{R}$, is a strong global solution of the SDE* (4.1.1).
2. *$\phi_{s,t}(x,\omega)$ is continuous in $(s,t,x) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$.*
3. *$\phi_{s,t}(\cdot,\omega) = \phi_{u,t}(\cdot,\omega) \circ \phi_{s,u}(\cdot,\omega)$ for all $s,u,t \in \mathbb{R}$.*
4. *$\phi_{s,s}(x,\omega) = x$ for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$.*
5. *$\phi_{s,t}(\cdot,\omega) : \mathbb{R}^d \to \mathbb{R}^d$ are homeomorphisms for all $s,t \in \mathbb{R}$.*

First, a suitable class of weighted Sobolev spaces must be introduced. Fix $p \in (1,\infty)$ and let $w : \mathbb{R}^d \to (0,\infty)$ be a Borel measurable function satisfying

$$\int_{\mathbb{R}^d} (1+|x|^p)w(x)dx < \infty.$$

Let $L^p(\mathbb{R}^d, w)$ denote the Banach space of all Borel measurable functions $u = (u_1,\dots,u_d) : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |u(x)|^p w(x)dx < \infty.$$

Moreover, denote by $W^{1,p}(\mathbb{R}^d, w)$ the linear space of functions $u \in L^p(\mathbb{R}^d, w)$ with weak partial derivatives $D_j u \in L^p(\mathbb{R}^d, w)$ for $j = 1,\dots,d$.

The first main result is stated in the next theorem. I refer to Section 2 in [5] for the quite involving proof.

Theorem 4.1.2. *In the SDE* (4.1.1)*, assume that the drift coefficient b is Borel-measurable and bounded. Then the SDE* (4.1.1) *has a Sobolev differentiable stochastic flow $\phi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$, $s,t \in \mathbb{R}$, i.e.*

$$\phi_{s,t}(\cdot) \text{ and } \phi_{s,t}^{-1}(\cdot) \ \in L^2(\Omega, W^{1,p}(\mathbb{R}^d, w))$$

for all $s,t \in \mathbb{R}$ and all $p \in (1,\infty)$.

The next part concerns the existence and uniqueness of solutions of the STE (4.1.2). The following notion of weak solution is used in [5].

Definition 4.1.3. *Let b be bounded and measurable and $u_0 \in L^\infty(\mathbb{R}^d)$. A weak solution of the transport equation (4.1.2) is a stochastic process $u \in L^\infty(\Omega \times [0,1] \times \mathbb{R}^d)$ such that, for every t, the function $u(t,\cdot)$ is weakly differentiable a.s. with*

$$\sup_{0 \le s \le 1, x \in \mathbb{R}^d} E[|Du(s,x)|^4] < \infty$$

and for every test function $\theta \in C_0^\infty(\mathbb{R}^d)$, the process $\int_{\mathbb{R}^d} \theta(x) u(t,x) dx$ has a continuous modification which is an $\mathcal{F}_t$-semi-martingale and

$$\begin{aligned}\int_{\mathbb{R}^d} \theta(x) u(t,x) dx = &\int_{\mathbb{R}^d} \theta(x) u_0(x) dx \\ &- \int_0^t \int_{\mathbb{R}^d} Du(s,x) \cdot b(s,x) \theta(x) dx ds \\ &+ \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s,x) D_i \theta(x) dx \right) \circ dB_s^i,\end{aligned} \tag{4.1.3}$$

where $Du(t,x)$ is the weak derivative of $u(t,x)$ in the space-variable.

Now for the second main result Theorem 20 in [5].

Theorem 4.1.4. *Let b be bounded and Borel measurable. Suppose $u_0 \in C_b^1(\mathbb{R}^d)$. Then there exists a unique weak solution $u(t,x)$ to the STE (4.1.2). For each $t > 0$ and all $p \in (1,\infty)$, the weak solution $u(t,\cdot)$ belongs a.s. to the weighted Sobolev space $W^{1,p}(\mathbb{R}^d, w)$. Moreover, for fixed t and x, $u(t,x)$ is Malliavin-differentiable.*

Sketch of proof. I here give a sketch of the proof for the existence of a weak solution. The idea is to use a uniformly bounded sequence of smooth functions $b_n : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ with compact support converging almost everywhere to b. Then there exists a unique strong solution $u_n(t,x) = u_0(\phi_{n,t}^{-1}(x))$, $n \ge 1$ to the STE (4.1.2) when b is replaced by b_n. Here $\phi_{n,t}$ denotes the flow of the SDE (4.1.1) driven by the vector field b_n. The process u_n is a differentiable weak L^∞-solution, such that for every $\theta \in C^\infty(\mathbb{R}^d)$

$$\begin{aligned}\int_{\mathbb{R}^d} \theta(x) u_n(t,x) dx = &\int_{\mathbb{R}^d} \theta(x) u_0(x) dx \\ &- \int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b_n(s,x) \theta(x) dx ds \\ &+ \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u_n(s,x) D_i \theta(x) dx \right) dB_s^i \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u_n(s,x) \Delta \theta(x) dx ds.\end{aligned} \tag{4.1.4}$$

Let $u(t,x) := u_0(\phi_t^{-1}(x))$ so that $u \in L^\infty(\Omega \times [0,1] \times \mathbb{R}^d)$, and $u(t,\cdot)$ is weakly differentiable almost surely. The goal is to show that $u(t,x)$ is a solution to the transport equation when n tends to infinity.

By Lemma 22 in [5] and dominated convergence, the next two limits exist in $L^2(\Omega)$

$$\int_{\mathbb{R}^d} \theta(x) u_n(t,x) dx \to \int_{\mathbb{R}^d} \theta(x) u(t,x) dx$$

$$\int_0^t \int_{\mathbb{R}^d} u_n(s,x) \Delta\theta(x) dx ds \to \int_0^t \int_{\mathbb{R}^d} u(s,x) \Delta\theta(x) dx ds.$$

By the Itô isometry

$$\sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u_n(s,x) D_i\theta(x) dx \right) dB_s^i \to \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s,x) D_i\theta(x) dx \right) dB_s^i$$

in $L^2(\Omega)$. The last limit

$$\int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot b_n(s,x) \theta(x) dx ds \to \int_0^t \int_{\mathbb{R}^d} Du(s,x) \cdot b(s,x) \theta(x) dx ds$$

in $L^2(\Omega)$, because all the other terms in Equation (4.1.4) converge.

To prove weak convergence of the last limit, it is a good idea to write the differences in three parts.

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} & Du_n(s,x) \cdot b_n(s,x) \theta(x) dx ds - \int_0^t \int_{\mathbb{R}^d} Du(s,x) \cdot b(s,x) \theta(x) dx ds \\
&= \int_0^t \int_{\mathbb{R}^d} Du_n(s,x) \cdot [b_n(s,x) - b(s,x)] \theta(x) dx ds \\
&+ \int_0^t \int_{\mathbb{R}^d} [Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))] D\phi_{n,s}^{-1}(x) \cdot b(s,x) \theta(x) dx ds \\
&+ \int_0^t \int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x)) [D\phi_{n,s}^{-1}(x) - D\phi_s^{-1}(x)] \cdot b(s,x) \theta(x) dx ds.
\end{aligned}$$

The first difference converges to zero by Hölder's inequality, Fubini's theorem, Proposition 7 in [5] and dominated convergence.

The second difference converges to zero by estimates that are consequences of Hölder's inequality and dominated convergence. See the proof in [5] for the details.

For the last term, let $X \in L^2(\Omega)$ and consider

$$\int_0^t E\left[\int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x)) (\phi_{n,s}^{-1}(x) - D\phi_s^{-1}(x)) \cdot b(s,x) \theta(x) X dx \right] ds.$$

Since for each s, Du_0, b and θ are bounded and $D\phi_s^{-1}$ is the weak limit of $D\phi_{n,s}^{-1}$, this expression tends to zero as $n \to \infty$. $\square$

Theorem 4.1.4 ensures that the solutions of the STE (4.1.2) will not "jump away" from each other, i.e. they do not explode.

4.2 Simulation of Solutions

In this section I present a simulation method for solutions of the stochastic transport equation (STE) in the 1-dimensional case. The method used in this thesis is based on the concept of brackets of stochastic processes as introduced in Eisenbaum [1].

Finally, I also aim at discussing the small noise problem of the STE with singular coefficients to obtain convergence to a Markov-selection-solution of ill-posed deterministic transport equation with irregular coefficients, by using the above mentioned simulation technique.

The goal of this section is to simulate solutions of the 1-dimensional stochastic transport equation

$$\begin{aligned} \partial_t u(t,x) + b(t,x)\partial_x u(t,x) + \partial_x u(t,x) \circ dB_t = 0, \\ u(0,x) = u_0(x). \end{aligned} \tag{4.2.1}$$

Here $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ and the Stratonovich integral $\partial_x u(t,x) \circ dB_t$ is the added multiplicative noise in the form of 1-dimensional Brownian motion. The drift coefficient $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ is merely bounded and measurable.

The choice of Stratonovich integral in Equation (4.2.1) is explained by Flandoli et al. [2] to have two reasons. First, (4.2.1) has an explicit solution given by $u(t,x) = u_0(\phi_t^{-1}(x))$. Here $\phi_t(x)$ is the flow map given by the unique strong solution $\{X_t^x\}_{t\geq 0}$ of the SDE

$$dX_t^x = b(t, X_t^x) + dB_t, \quad t \geq 0, \ X_0^x = x.$$

The other reason is motivated by the Wong-Zakai principle, see [p 3 in 2] for details.

As shown by Mohammed et al. [5], this gives that $(x \mapsto X_t^x) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$. Hence, X_t^x is differentiable in x and the derivative is given by

$$\frac{\partial}{\partial x} X_t^x = \exp\left(\int_0^t \int_{\mathbb{R}} b(s,y) L^{X^x}(ds,dy) \right) = \exp\left([b(\cdot, X^x), X^x]_t \right), \tag{4.2.2}$$

which holds if $(s \mapsto b(s,\cdot))$ is continuous in s as a map from $[0,T]$ to $L^2_{\mathrm{loc}}(\mathbb{R})$. Here L^{X^x} is the local time measure. The expression

$$[b(\cdot, X^x), X^x]_t := \lim_{n\to\infty} \sum_{0<t_1<\cdots<t_n\leq t} [b(t_i, X^x_{t_{i+1}}) - b(t_i, X^x_{t_i})](X^x_{t_{i+1}} - X^x_{t_i}), \tag{4.2.3}$$

converges uniformly in probability and can be found in Eisenbaum [1].

By the definition of the stochastic flow of diffeomorphisms (see e.g. Definition 1 (a) in [2]) $\phi_t(x) = X_t^x$. Since $\phi_t : \mathbb{R} \to \mathbb{R}$ is bijective, the values for all y are given by the derivative

$$(\phi_t^{-1})'(y) = \frac{1}{\phi_t'(x)} = \frac{1}{\partial_x X_t^x}.$$

Hence, $\phi_t^{-1}(y)$ can be found by integrating the inverse derivative of X_t^x with respect to x.

The simulation of the solutions to the STE (4.2.1) is performed using the following six-step algorithm.

1. **Simulation of the paths of X^x for continuous and bounded b:**
   Let $\Delta_t = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ denote a partition of $[0, T]$ where $t_i := i\Delta t$ for $i = 0, \ldots, n$ and $\Delta t = T/n$. Let $X_i^x = X_{t_i}^x$. Then Equation (4.2.1) can be approximated by
   $$X_{i+1}^x = X_i^x + b(t_i, X_i^x)\Delta t + \Delta B_i, \tag{4.2.4}$$
   where $\Delta B_i = B_{i+1} - B_i = \sqrt{\Delta t} g_i$ and $g_i \sim N(0, 1)$ iid. Sample $g_0^*, \ldots, g_n^*$ from the standard normal distribution and use them to compute X_i^x in Equation (4.2.4) for different starting values x.

2. **Approximation of Equation (4.2.2):**
   Insert the values of X_t^x into the expression
   $$[b(\cdot, X^x), X^x]_t \approx \sum_{t_{i+1} \leq t} [b(X_{t_{i+1}}^x, t_i) - b(X_{t_i}^x, t_i)](X_{t_{i+1}}^x - X_{t_i}^x)$$
   for different x and $t \in \Delta_t$. Then the derivative
   $$\partial_x X_t^x \approx \exp\left(\sum_{t_{i+1} \leq t} [b(t_i, X_{t_{i+1}}^x) - b(t_i, X_{t_i}^x)](X_{t_{i+1}}^x - X_{t_i}^x) \right)$$
   for different x, t.

3. **Calculation of the inverse $f^{-1}(y)$ of $f(x) := X_t^x$:**
   From the relation
   $$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{\partial_x X_t^x} \approx \exp\left(- \sum_{t_{i+1} \leq t} [b(X_{t_{i+1}}^x, t_i) - b(X_{t_i}^x, t_i)](X_{t_{i+1}}^x - X_{t_i}^x) \right)$$
   for all x, we obtain $(f^{-1})'(y)$ for all y since $f : \mathbb{R} \to \mathbb{R}$ is bijective.

4. **Integration of $(f^{-1})'$:**
   $f^{-1}(y) = f^{-1}(y_0) + \int_{y_0}^{y} (f^{-1})'(u)du = x^* + \int_{y_0}^{y} (f^{-1})'(u)du$, where $f(x^*) = y_0$, for different y, t. For the purpose of integration I use the trapezoid method.

5. **Computation of the first solution path of the STE:**
   Insert $f^{-1}(y)$ into u_0, e.g. $u_0(f^{-1}(y))$.

6. **Additional solution paths:**
   Repeat 1-5 in order to generate additional solution paths.

The candidate for the drift coefficient
$$b(t, x) = 2\,\mathrm{sign}(x)\sqrt{|x|}, \tag{4.2.5}$$
where
$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

This gives an error bound of $O(\Delta t)$ in the L^2-sense.

In the simulation of the solutions to the STE (4.2.1) and the small noise simulations, I have used the interval $[-0.2,\ 0.2]$ with step length $\Delta x = 0.02$ for the starting values x. The time interval is $[0, 1]$ with step length $\Delta t = 0.01$.

In Figure 4.2 six solutions to the STE (4.2.1) with $u_0(x) = \cos(f^{-1}(x))$ are shown. They all look very irregular and differ a great deal from each other.

In Figure 4.4 and 4.3, I have plotted the solution to the stochastic transport equation (4.2.1) with $u_0 = \text{sign}(f^{-1})$.

There are six plots in each figure corresponding to a weaker and weaker Brownian noise. The noise is muffled by a factor $\epsilon = (1,\ 2^{-1},\ 2^{-2},\ 2^{-3},\ 2^{-4},\ 2^{-5})$. The plots show that the simulated solutions converge to a certain solutions of the deterministic transport equation, which has infinitely many solutions. Such a solution is called a Markov selection and can be interpreted as a stable solution in sense of perturbation by small noise.

The theory predicts that the simulated paths cannot diverge from each other because of Malliavin differentiability.

4.3 Closing Thoughts

Adding noise to PDE's that have no unique solutions in order to obtain a unique solution is an exciting idea. One might be tempted to contemplate what might happen if this was done with the Einstein field equation (EFE), which is a system of ten coupled, non-linear, hyperbolic-elliptic PDE's. The EFE describes the space-time geometry under the influence of mass-energy an linear momentum. These equations only have exact solutions under simplifying assumptions. The question whether one were able to add noise to the manifold to make it rough in order to obtain regularity in the solutions, could be an interesting topic for further investigations.

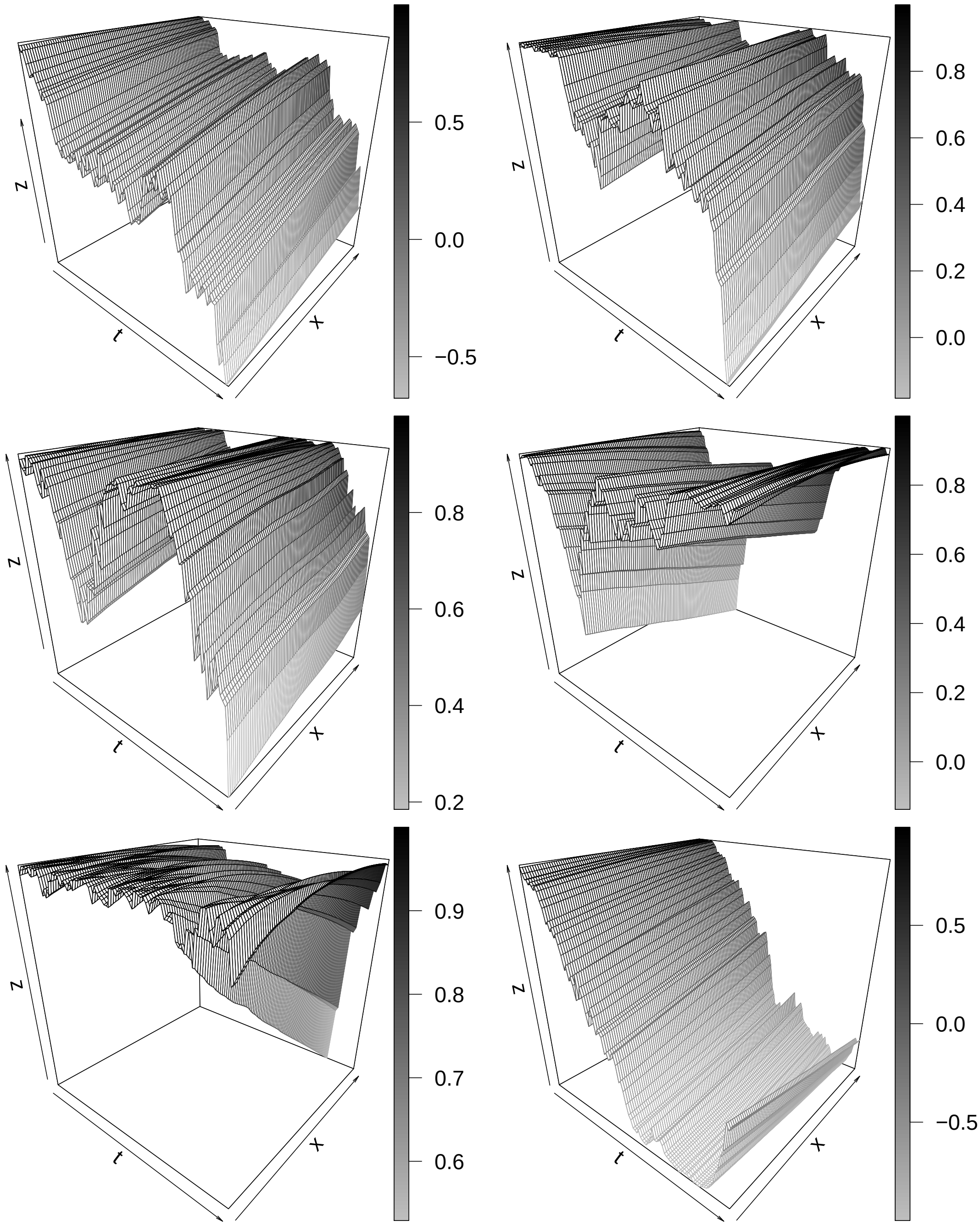
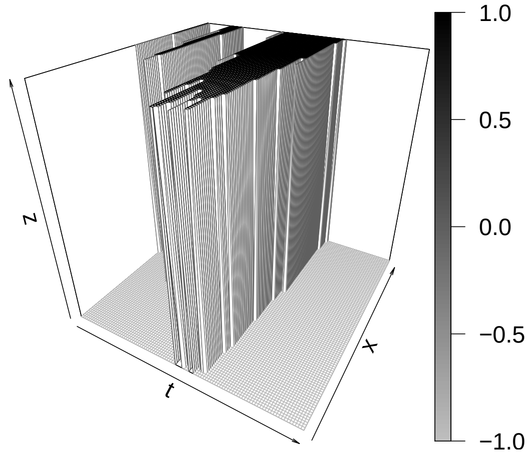
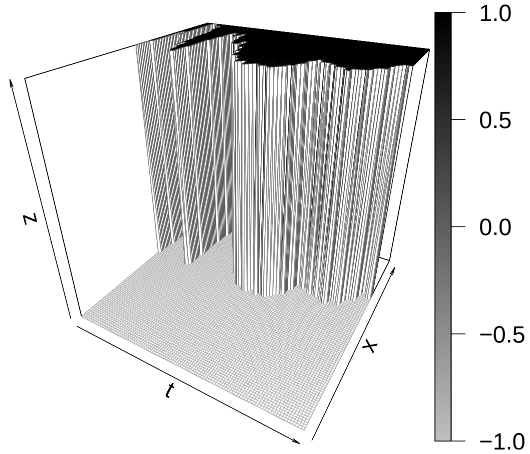


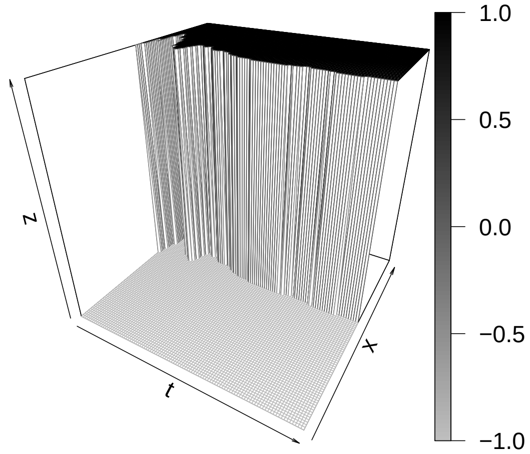
Figure 4.2: Six different solutions to the STE for a sample path X_t^x initial function $u_0(x) = \cos(x)$.



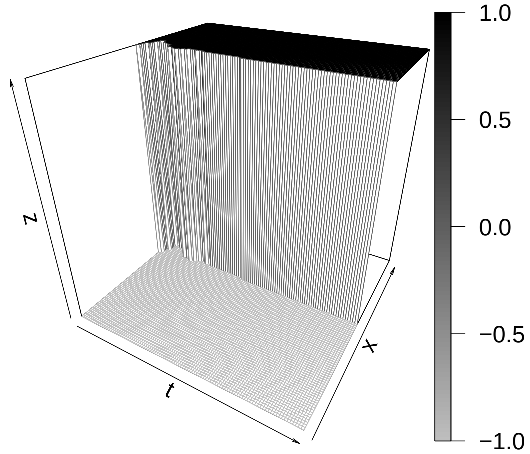
(a) $\epsilon = 1$



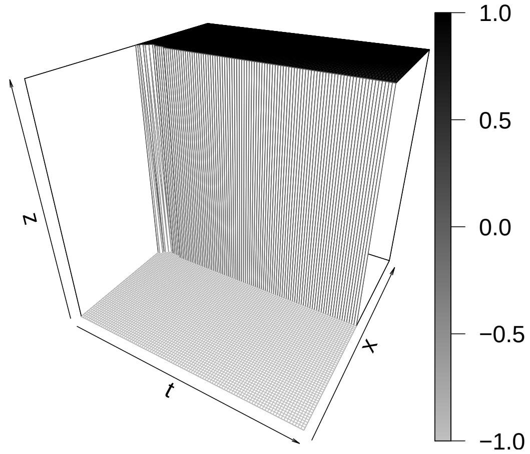
(b) $\epsilon = 2^{-1}$



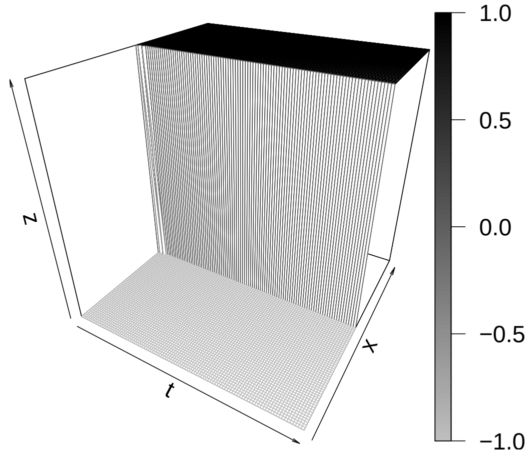
(c) $\epsilon = 2^{-2}$



(d) $\epsilon = 2^{-3}$

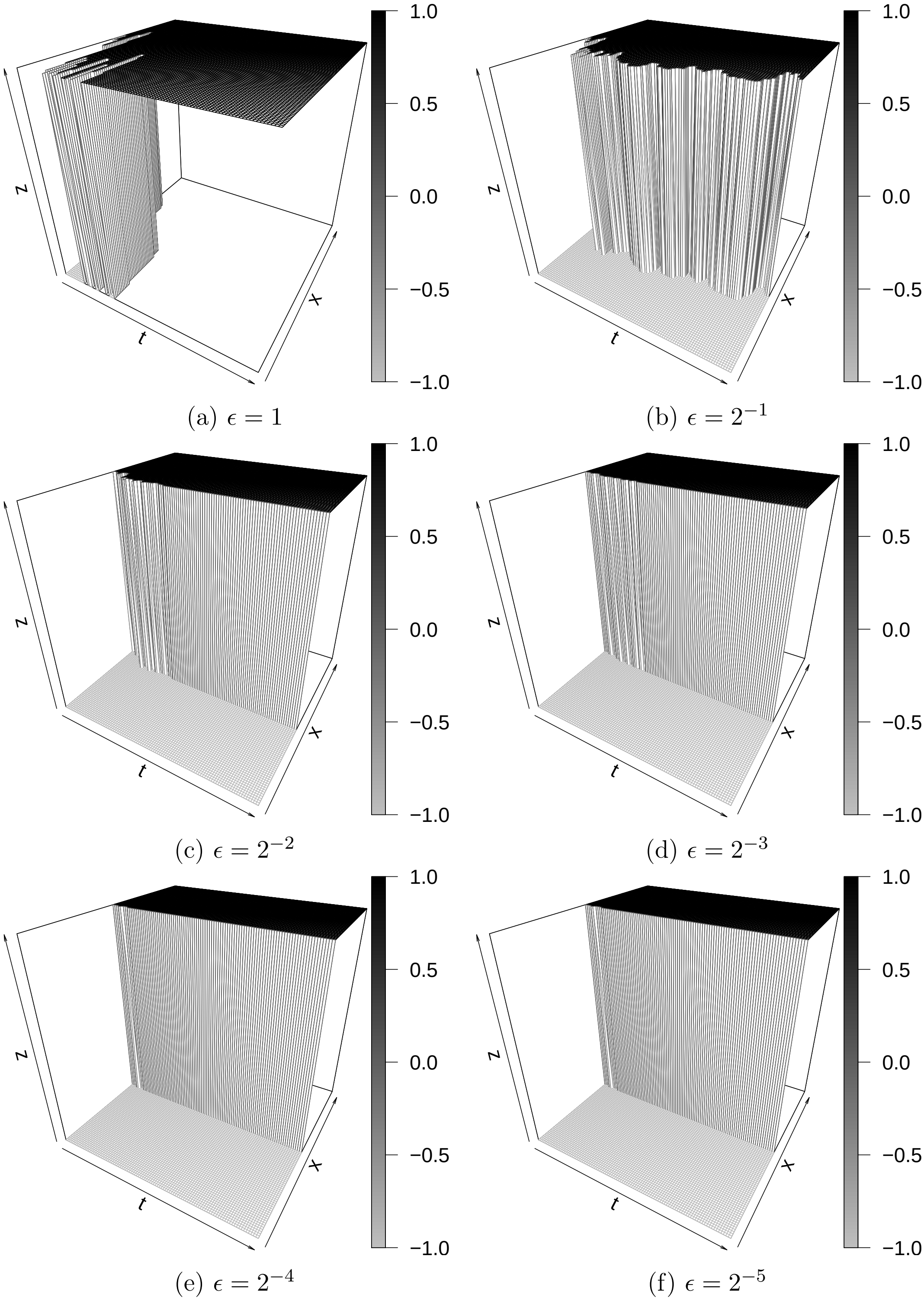


(e) $\epsilon = 2^{-4}$



(f) $\epsilon = 2^{-5}$

Figure 4.3: Solution to the STE for a sample path X_t^x with vanishing Brownian noise.



(a) $\epsilon = 1$
(b) $\epsilon = 2^{-1}$
(c) $\epsilon = 2^{-2}$
(d) $\epsilon = 2^{-3}$
(e) $\epsilon = 2^{-4}$
(f) $\epsilon = 2^{-5}$

Figure 4.4: Solution to the STE for a sample path X_t^x with vanishing Brownian noise.

Appendix

This appendix contains the R code used to simulate the solutions of the STE (4.2.1).

I start with some functions.

```
sgn < function(x){
    ## Signum function. Returns a vector of length=length(x), with  1 for
        elements where x<0 and 1 else.
    ifelse(x<0,  1,1)
}

b < function(x, case){
    ## Non lipschitz drift function in transport equation.
    if (case == "cont" || case == "non.mall"){
        return(2*sgn(x)*sqrt(abs(x)))
    }
    if (case == "non.cont"){
        return(abs(x))
    }
}

u0 < function(x, case){
    ## Initial function for SDE
    if (case == "cont" || case == "non.cont"){
        return(cos(x))
    }
    if (case == "non.mall"){
        return(sgn(x))
    }
}

dXdx < function(t,d1,d2){
    ## Function to calculate approximation to (++) in step 2 from notes.
    exp(sum(d1[1:t]*d2[1:t]))
}

trapz < function(dx,y){
    ## This function calculates the integral of a linearly interpolated
        function by the trapezoid method.
    sum(dx*(y[ length(y)]+y[ 1]))/2
}

init.vars < function(xsteps = 100, x.int = c( .5,.5), tsteps=100, paths
    =1, t.end=1, n.eps = 1, case="cont"){
    ## Assigning variables to be axcessed globally. Default is 101 x steps
         in the interval [ .5,.5], 101 time steps in the interval [0,1],
        one solution path, eps = 1.
    T < t.end
    assign("N", paths, envir = .GlobalEnv)                ## Number of paths
    assign("n", xsteps, envir = .GlobalEnv)               ## Number of x
        steps
    assign("x0", seq(x.int[1],x.int[2],length.out = n+1), envir = .
        GlobalEnv)
```

```
    assign("n.t", tsteps, envir = .GlobalEnv)           ## Number of time/
        space points
    assign("delta.t", T/n.t, envir = .GlobalEnv)        ## Time step length
    g < matrix(rnorm(N*(n.t+1)), ncol = N)
    sqrt.dtg < sqrt(delta.t)*g                           ## "Brownian motion
         steps"
    assign("eps",  2^( (0:(n.eps 1))), envir = .GlobalEnv)
    assign("eps.dB", sqrt.dtg%o%eps, envir = .GlobalEnv) ## eps times BM
    assign("case", case, envir = .GlobalEnv)
}
```

The next code block is the routine for the algorithm in Section 4.2.

```
transport.sim < function(eps.dB, x0, case){
    ## This function returns the solution to the SDE (4.1.). It takes a
        vector of simulated Brownian motion steps multiplied by a factor
        eps in [0,1], and a vector of starting x values. The case parameter
         is passed to functions.
    X < rep(1,n.t+1)%o%x0  ## Matrix where each row corresponds to fixed
        t and each column to fixed x

    ## Step 1: Simulate a solution of the SDE (4.1.)
    for (i in 1:n.t){
        X[i+1,] < X[i,] + b(X[i,],case)*delta.t + eps.dB[i]
    }

    ## Step 2: Compute the derivatives given by eq. (4.1.)
    delta.Xt < X[2:(n.t+1),] X[1:n.t,]
    delta.b < b(X[2:(n.t+1),],case) b(X[1:n.t,],case)

    ## matr.dXdx is a matrix containing the derivatives dX/dx of with the
        fixed t for each row and fixed x for each column
    matr.dXdx < sapply(1:(n+1), function(x) sapply(1:n.t, function(t)
        dXdx(t,delta.b[,x], delta.Xt[,x])))

    ## Step 3: Compute the inverse of the derivative
    inv.dfdx < 1/matr.dXdx

    ## Step 4: Integrate inv.dfdx over y=X, so over fixed t, i.e. over
        each row
    delta.Xx < X[,2:(n+1)] X[,1:n]
    inv.f < sapply(2:(n+1), function(x) sapply(1:n.t, function(t) trapz(
        delta.Xx[t,1:(x 1)],inv.dfdx[t,1:x])+X[t,1]))

    ## Return the value
    inv.f
    ## u0(inv.f,case)
}
```

Now comes the part of the code where the functions are run. First the simulation of the solutions in Figure 4.2. Then a the simulations for the small noise problem.

```
## Initialise variables: initial x values [ 0.2,0.2], 6 solution paths
init.vars(x.int = c( .2,.2), paths = 6)
```

```
## Run the simulation
sim < sapply(1:length(eps), function(eps) sapply(1:N, function(path)
    transport.sim(eps.dB[,path,eps], x0, case), simplify = "array"),
    simplify = "array")

## View the surface
par(mfrow = c(3,2), mar = c(.5,.5,.5,1.5))

require(plot3D)

for (i in 1:N){
     persp3D(z= sim[,,i,2], xlab="t", ylab="x", colkey = list(length=.8,
         dist= .04), theta = 30, phi = 30, col = ramp.col(), border = NA,
         facets = FALSE)
}

## ******************
## Small noise problem
## ******************

## Initialise variables: initial x values [0.2,0.2], eps = (1, 0.5, 0.25,
     0.125, 0.0625, 0.03125), 10 solution paths
init.vars(x.int = c( .2,.2), n.eps = 6, paths = 10)

## Run the simulation
sim2 < sapply(1:length(eps), function(eps) sapply(1:N, function(path)
    transport.sim(eps.dB[,path,eps], x0, case), simplify = "array"),
    simplify = "array")

## View the surface
par(mfrow = c(3,2), mar = c(.5,.5,.5,1.5))

require(plot3D)

for (j in 1:N){
     for (i in 1:length(eps)){
     persp3D(z= sim2[,,j,i], xlab="t", ylab="x", colkey = list(length=.8,
         dist= .04), theta = 40, phi = 25, col = ramp.col(), border = NA,
         facets = FALSE)
     }
     readline("Press [Enter] to continue... ")
}
```

Bibliography

[1] N. Eisenbaum. Integration with respect to local time. *Potential Analysis*, 2000.

[2] F. Flandoli, E. Gubinelli, and E. Peiola. Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.*, 2010.

[3] Ionnis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus.* Springer-Verlag, 1991.

[4] Hui-Hsiung Kuo. *Introduction to Stochastic Integration.* Springer, 2006.

[5] S.E.A. Mohammed, T. Nielsen, and F. Proske. Sobolev differential stochastic flows for sde's with singular coefficients: Application to transport equation. *Annals of Probability*, 2014.

[6] Giulia Di Nunno, Bernt Øksendal, and Frank Proske. *Malliavin Calculus for Lévy Processes with Applications to Finance.* Springer, 2009.

[7] A.K. Zvonkin. A transformation of the phase space of a diffusion process that removes the drift. *Math. USSR*, 1974.

[8] Bernt Øksendal. *Stochastic Differential Equations: An Introduction with Applications.* Springer, 6th edition, 2003.